On Burr III-Pareto Distribution: Development, Properties, Characterizations and Applications

Fiaz Ahmad Bhatti
National College of Business Administration and Economics,
Lahore Pakistan
fiazahmad72@gmail.com

G.G. Hamedani
Marquette University, Milwaukee,
WI 53201-1881, USA
g.hamedani@mu.edu

Mustafa Ç. Korkmaz
Department of Measurement and Evaluation,
Artvin Çoruh University, Artvin, Turkey
mcagatay@artvin.edu.tr ; mustafacagataykorkmaz@gmail.com

Munir Ahmad
National College of Business Administration and Economics,
Lahore Pakistan
munirahmaddr@yahoo.co.uk

Abstract

In this paper, a new four parameter lifetime model with increasing, decreasing, increasing-decreasing, decreasing-increasing-decreasing, modified bathtub, bathtub and inverted bathtub hazard rate function called Burr III-Pareto (BIII-Pareto) is developed on the basis of the T-X family technique. The BIII-Pareto density function is arc, J-shape, reverse J-shape, positively, negatively skewed and symmetrical. Some structural and mathematical properties including moments, moments of order statistics, inequality measures and reliability measures are theoretically established. The BIII-Pareto distribution is characterized via different techniques. Parameters of the BIII-Pareto distribution are estimated using maximum likelihood method. The simulation study for performance of the maximum likelihood estimates (MLEs) of parameters for the BIII-Pareto distribution is carried out. The potentiality of the BIII-Pareto distribution is demonstrated by its application to real data sets. Goodness of fit of this distribution through different methods is studied. The BIII-Pareto distribution is empirically better for lifetime applications.

Keywords: Moments, Reliability, Characterizations, Maximum Likelihood Estimation.

1. Introduction

The Pareto model (Pareto; 1896) was introduced for unequal income distribution. The Pareto distribution is suitable for different areas of research such as insurance, finance, reliability and extreme weather.

During recent decades, many continuous univariate distributions have been developed but various data sets from reliability, engineering, environmental, financial, biomedical sciences, among other areas, do not follow these distributions. Therefore, modified, extended and generalized distributions and their applications to problems in these areas is a clear need of day.

The modified, extended and generalized distributions are obtained by the introduction of some transformation or addition of one or more parameters to the baseline distribution. These new developed distributions provide better fit to data than their sub-models.

Many modified, extended and generalized forms of the Pareto distribution are available in the literature such as generalized Pareto (Pickands;1975), exponentiated Pareto (Stoppa;1990), exponentiated Pareto using Lehmann alternative type1(Gupta et al.;1998), generalized Pareto (Choulakian and Stephens; 2001), generalized Pareto (Pisarenko and Sornette; 2003), Pareto model for OLAP (Nadeau and Teorey; 2003), extended Pareto (Akinsete et al. ;2008), beta-Pareto (Eugene et al.;2002), Pareto (Farshchian and Posner; 2010), beta-EP (Nassar and Nada ; 2011), beta-G Pareto (Zea et al.;2012 and Mansoor ;2013), beta generalized Pareto (Mahmoudi; 2011), gamma- Pareto (Alzaatreh et al., 2011, 2013b), Weibull-Pareto (Alzaatreh et al. ;2013a), Kumaraswamy-Pareto (Bourguignon et al.;2013), new Weibull Pareto (Nasiru and Luguterah; 2015), Pareto(Arnold;2015), new Pareto (Bourguignon et al.;2016), Weibull–Pareto (Tahir et. al.;2016) and generalized Weibull Pareto distribution (Isah and Bala; 2017).

The main purpose of this article is to obtain a more flexible distribution for the lifetime applications called the BIII-Pareto distribution. The BIII-Pareto density is arc, J-shape, reverse J-shape, positively, negatively skewed and symmetrical. The BIII-Pareto distribution has, increasing, decreasing, increasing-decreasing, decreasing-increasing-decreasing, modified bathtub, bathtub and inverted bathtub hazard rate function.

The flexible nature of the hazard rate function of the BIII-Pareto distribution helps to serve as the best alternative model to the current models for modeling real data in economics, life testing, reliability, survival analysis and other related areas of research. The BIII-Pareto distribution provides better fit than sub-models.

Our interest is to study the BIII-Pareto distribution in terms of its mathematical properties, applications and comparison to sub-models.

The article is composed of the following sections. In Section 2, the BIII-Pareto distribution is development on the basis of the T-X family technique. In Section 3, the BIII-Pareto distribution is studied in terms of the basic structural properties, sub-models; descriptive measures based quantiles and plots. In Section 4, moments, moments of order statistics, incomplete moments, inequality measures, residual life function and some other properties are presented. In Section 5, reliability measures are studied. In Section 6, the BIII-Pareto distribution is characterized via (i) conditional expectation; (ii) ratio of truncated moments; (iii) reverse hazard rate function; (iv) elasticity function; (v) conditional expectation of certain function of the random variable; (vi) conditional expectation of lower record values and (vii) conditional expectation of lower record values with spacing. In Section 7, the parameters of BIII-Pareto are estimated using maximum likelihood method. In Section 8, the simulation study for the performance of the MLEs of the BIII-Pareto distribution with respect to sample size n is carried out. In Section 9, the potentiality of the BIII-Pareto distribution is demonstrated by its application to real data sets. Goodness of fit of the probability distribution through different methods is studied. The concluding remarks are given in Section 10.

2. DEVELOPMENT OF BIII-PARETO DISTRIBUTION

The probability density function (pdf) and cumulative distribution function (cdf) of the Pareto distribution are given, respectively, by

$$g(x;\kappa,\theta)=\frac{\kappa}{\theta}\left(\frac{x}{\theta}\right)^{-\kappa}, \quad x>\theta, \kappa>0, \theta>0 \tag{1}$$

and

$$G(x;\kappa,\theta)=1-\left(\frac{x}{\theta}\right)^{-\kappa}, \quad x\geq\theta, \kappa>0, \theta>0. \tag{2}$$

The odds ratio for the Pareto random variable X is

$$W(G(x))=\frac{G(x;\kappa,\theta)}{\bar{G}(x;\kappa,\theta)}=\frac{1-\left(\frac{x}{\theta}\right)^{-\kappa}}{\left(\frac{x}{\theta}\right)^{-\kappa}}=\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right].$$

Gurvich et al. (1997) replaced 'x' with odds ratio in the Weibull distribution for the development of a class of extended Weibull distributions. Alzaatreh et al. (2013) developed the cdf of the T-X family of distributions as

$$F(x)=\int_{a}^{W(G(x))} r(t)\,dt,$$

where $W(G(x))$ is a function of $G(x)$ and $r(t)$ is the pdf of a non-negative random variable.

Bourguignon et al. (2014) inserted the odds ratio of a baseline distribution in place of 'x' in the cdf of the Weibull distribution for the development of a new family of distributions.

The BIII-Pareto is developed by inserting the odds ratio of the Pareto random variable in place of 'x' in the cdf of the BIII distribution. The cdf for BIII-Pareto distribution is obtained as

$$F(x)=\int_{0}^{W(G(x))}\alpha\beta t^{-\beta-1}\left(1+t^{-\beta}\right)^{-\alpha-1}dt,$$

or

$$F(x;\alpha,\beta,\kappa,\theta)=\int_{0}^{\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]}\alpha\beta t^{-\beta-1}\left(1+t^{-\beta}\right)^{-\alpha-1}dt,$$

or

$$F(\mathrm{x})=\left\{1+\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\alpha}, \alpha>0, \beta>0, \kappa>0, x\geq\theta. \tag{3}$$

The pdf of BIII-Pareto distribution is

$$f(x)=\alpha\beta\frac{\kappa}{\theta}\left(\frac{x}{\theta}\right)^{\kappa-1}\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta-1}\left\{1+\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\alpha-1}, \quad x>\theta. \tag{4}$$

For $\alpha=1$, the BIII-Pareto distribution reduces to Log-logistic-Pareto (LL-Pareto) and for $\beta=1$, the BIII-Pareto distribution reduces to inverse Lomax-Pareto (IL-Pareto).

2.1 Transformations and Compounding

In this sub-section, the BIII-Pareto distribution is derived through (i) ratio of the exponential and gamma random variables and (ii) the generalized inverse Weibull Pareto (GIWP) and gamma distributions.

L**emma (i)** *If* $Z_1 \sim \exp(1), Z_2 \sim gamma(\alpha, 1)$, then for $Z_1=\left[\left(\frac{X}{\theta}\right)^{\kappa}-1\right]^{-\beta} Z_2$, we have

$$X=\theta\left[\left(\frac{Z_2}{Z_1}\right)^{\frac{1}{\beta}}+1\right]^{\frac{1}{\kappa}} \sim BIII-Pareto(\alpha,\beta,,\theta,\kappa).$$

(ii) If $X;\beta,\theta,\kappa|\eta \sim w(x;\beta,\theta,\kappa|\eta)=\eta\beta\frac{\kappa}{\theta}\left(\frac{x}{\theta}\right)^{\kappa-1}\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta-1} e^{-\eta\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}}$

and $\eta|\alpha \sim g(\alpha,\eta)=\frac{1}{\Gamma(\alpha)}\eta^{\alpha-1}e^{-\eta}, \eta>0$, then, the BIII-Pareto distribution is obtained via integrating the effect of η with the help of

$$f(x,\alpha,\beta,\theta,\kappa)=\int_0^{\infty} w(x;\beta,\theta,\kappa|\eta)g(\alpha,\eta)d\eta. \quad (5)$$

So $X \sim BIII-Pareto(\alpha,\beta,\theta,\kappa)$.

3. STRUCTURAL PROPERTIES OF BIII-PARETO DISTRIBUTION

The survival, hazard, cumulative hazard, reverse hazard functions and Mills ratio of a random variable X with BIII-Pareto distribution are given, respectively, by

$$S(x)=1-\left\{1+\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\alpha}, \quad (6)$$

$$h(x)=\frac{\alpha\beta\frac{\kappa}{\theta}\left(\frac{x}{\theta}\right)^{\kappa-1}\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta-1}\left\{1+\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\alpha-1}}{1-\left\{1+\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\alpha}},\ x>\theta, \quad (7)$$

$$H(x)=\ln\left\{1-\left\{1+\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\alpha}\right\}\ x>\theta, \quad (8)$$

$$\mathrm{r(x)}=\alpha\beta\frac{\kappa}{\theta}\left(\frac{x}{\theta}\right)^{\kappa-1}\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta-1}\left\{1+\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-1},\quad x>\theta, \quad (9)$$

and

$$m(x)=\frac{1-\left\{1+\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\alpha}}{\alpha\beta\frac{\kappa}{\theta}\left(\frac{x}{\theta}\right)^{\kappa-1}\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta-1}\left\{1+\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\alpha-1}},\ x>\theta. \tag{10}$$

The elasticity $e(x)=\frac{d\ln \mathrm{F}(x)}{d\ln x}=xr(x)$ of BIII-Pareto distribution is given by

$$e(x)=\alpha\beta\kappa\left(\frac{x}{\theta}\right)^{\kappa}\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta-1}\left\{1+\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-1},\ x>\theta. \tag{11}$$

The elasticity of the BIII-Pareto distribution shows the behavior of the accumulation of probability in the domain of the random variable.

3.1 Shapes of the BIII-Pareto Density and Hazard Rate Functions

The following graphs show that shapes of the BIII-Pareto density are arc, J-shape, reverse J-shape, positively, negatively skewed and symmetrical. The BIII-Pareto distribution has, increasing, decreasing, increasing-decreasing, decreasing-increasing-decreasing, modified bathtub, bathtub and inverted bathtub hazard rate function.

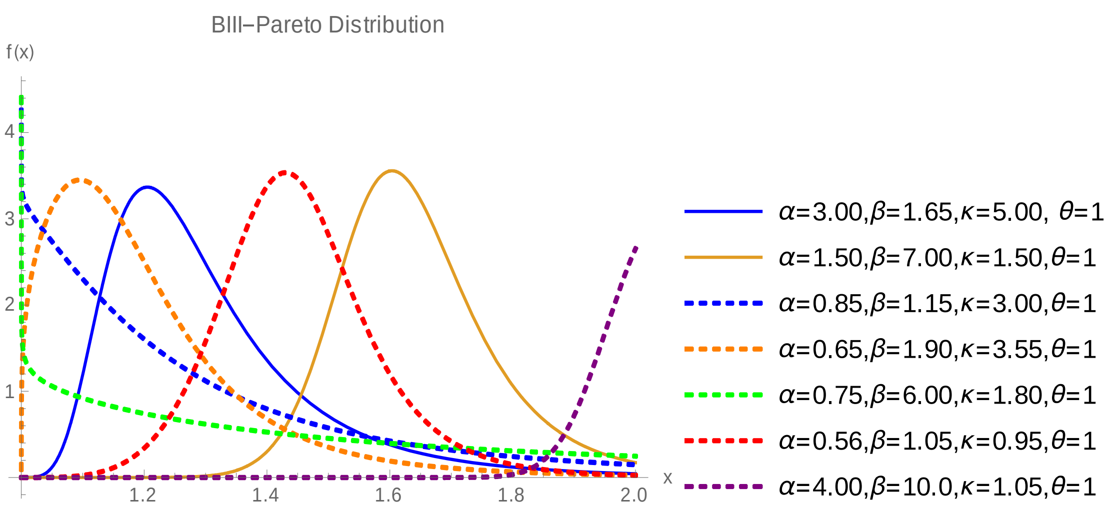

Fig. 1: Plots of pdf of the BIII-Pareto Distribution for the selected parameters values

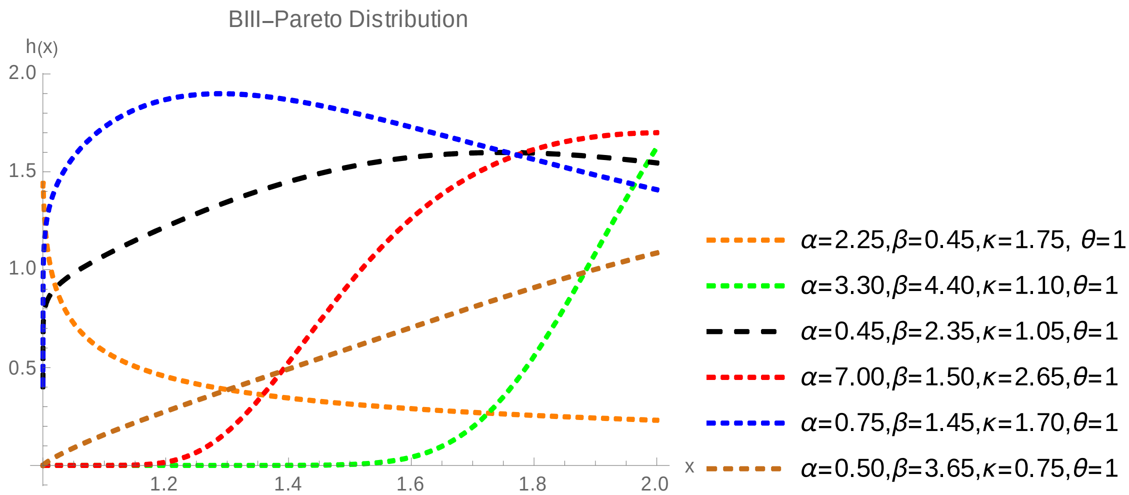

Fig. 2: Plots of hrf of the BIII-Pareto Distribution for the selected parameters values

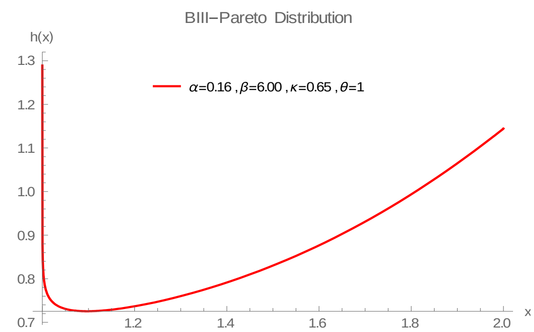

Fig. 3: Plot of hrf of the BIII-Pareto Distribution

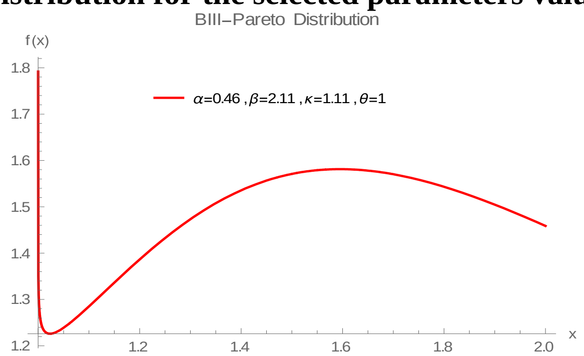

Fig. 4: Plot of hrf of the BIII-Pareto Distribution

3.2 Quantile function of the BIII-Pareto Distribution

The quantile function of the BIII-Pareto distribution is $x_q = \theta\left[\left(q^{-\frac{1}{\alpha}} - 1\right)^{-\frac{1}{\beta}} + 1\right]^{\frac{1}{\kappa}}$. The median of the BIII-Pareto distribution is $x_{med} = \theta\left[\left(2^{\frac{1}{\alpha}} - 1\right)^{-\frac{1}{\beta}} + 1\right]^{\frac{1}{\kappa}}$. The random number generator for BIII-Pareto distribution is

$X = \theta\left[\left(Z^{-\frac{1}{\alpha}} - 1\right)^{-\frac{1}{\beta}} + 1\right]^{\frac{1}{\kappa}}$, where the random variable Z has the uniform distribution on $(0,1)$.

Some measures based on quartiles for location, dispersion, skewness and kurtosis for the BIII-Pareto distribution respectively are: Median $M = Q_{\frac{1}{2}}$ and quartile deviation is

$Q.D.=\frac{Q_{\frac{3}{4}}-Q_{\frac{1}{4}}}{2}$ .Bowley's skewness measure is $S_q=\frac{Q_{\frac{3}{4}}-2Q_{\frac{1}{2}}+Q_{\frac{1}{4}}}{Q_{\frac{3}{4}}-Q_{\frac{1}{4}}}$ and Moors kurtosis measure based on octiles is $K=\frac{Q_{\frac{7}{8}}-Q_{\frac{5}{8}}+Q_{\frac{3}{8}}-Q_{\frac{1}{8}}}{Q_{\frac{6}{8}}-Q_{\frac{2}{8}}}$. The quantile based measures exist even for distributions that have no moments. The quantile based measures are less sensitive to the outliers.

4. MOMENTS

Moments, incomplete moments, inequality measures, residual life functions and some other properties are provided here.

4.1 Moments of the BIII-Pareto Distribution

The r[th] moment about the origin of X with the BIII-Pareto distribution is

$$\mu_r'=E(X^r)=\int_a^b x^r f(x)dx,$$

$$E\left(X^r\right)=\int_\theta^\infty x^r\alpha\beta\frac{\kappa}{\theta}\left(\frac{x}{\theta}\right)^{\kappa-1}\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta-1}\left\{1+\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\alpha-1}dx.$$

Let $\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}=w,\quad x=\theta\left(w^{-\frac{1}{\beta}}+1\right)^{\frac{1}{\kappa}},\quad x^r=\theta^r\left(w^{-\frac{1}{\beta}}+1\right)^{\frac{r}{\kappa}}$, then

$$E\left(X^r\right)=\alpha\theta^r\int_0^\infty\left(1+w^{-\frac{1}{\beta}}\right)^{\frac{r}{\kappa}}\{1+w\}^{-\alpha-1}dx,$$

$$E\left(X^r\right)=\alpha\theta^r\sum_{i=0}^{\frac{r}{\kappa}}\binom{\frac{r}{\kappa}}{i}\int_0^\infty w^{-\frac{i}{\beta}}\{1+w\}^{-\alpha-1}dx$$

$$\mu_r'=E\left(X^r\right)=\alpha\theta^r\sum_{i=0}^{\frac{r}{\kappa}}\binom{\frac{r}{\kappa}}{i}B\left(1-\frac{i}{\beta},\alpha+\frac{i}{\beta}\right),\ \mathrm{r}=1,2,3,\dots. \qquad (12)$$

Mean and Variance of the BIII-Pareto distribution are

$$E(X)=\alpha\theta\sum_{i=0}^{\frac{1}{\kappa}}\binom{\frac{1}{\kappa}}{i}B\left(1-\frac{i}{\beta},\alpha+\frac{i}{\beta}\right), \qquad (13)$$

$$\mathrm{Var}(X)=\left\{\alpha\theta^2\sum_{i=0}^{\frac{2}{\kappa}}\binom{\frac{2}{\kappa}}{i}B\left(1-\frac{i}{\beta},\alpha+\frac{i}{\beta}\right)-\left[\alpha\theta\sum_{i=0}^{\frac{1}{\kappa}}\binom{\frac{1}{\kappa}}{i}B\left(1-\frac{i}{\beta},\alpha+\frac{i}{\beta}\right)\right]^2\right\}. \qquad (14)$$

The factorial moments for the BIII-Pareto distribution are $E\left[X\right]_n = \sum_{r=1}^{n} \varphi_r E(X^r),$

$$E\left[X\right]_n = \sum_{r=1}^{n} \varphi_r \alpha\theta^r \sum_{i=0}^{\frac{r}{\kappa}} \binom{\frac{r}{\kappa}}{i} B\left(1-\frac{i}{\beta}, \alpha+\frac{i}{\beta}\right), \tag{15}$$

where $\left[X\right]_i = X\left(X+1\right)\left(X+2\right)......\left(X+i-1\right)$ and φ_r is Stirling number of the first kind.

The Mellin transform helps to determine moments for a probability distribution. By definition, the Mellin transform for the BIII-Pareto distribution is

$$\mathrm{M}\left\{f\left(x\right);\mathrm{s}\right\} = f^{*}\left(s\right) = \int_{0}^{\infty} f\left(x\right)\, x^{s-1}dx. \tag{16}$$

$$\mathrm{M}\left\{f\left(x\right);\mathrm{s}\right\} = \int_{\theta}^{\infty} x^{s-1}\alpha\beta\frac{\kappa}{\theta}\left(\frac{x}{\theta}\right)^{\kappa-1}\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta-1}\left\{1+\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\alpha-1} dx,$$

Let $\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta} = w$, then

$$\mathrm{M}\left\{f\left(x\right);\mathrm{s}\right\} = \alpha\theta^{(s-1)} \sum_{i=0}^{\frac{(s-1)}{\kappa}} \binom{\frac{(s-1)}{\kappa}}{i} B\left(1-\frac{i}{\beta}, \alpha+\frac{i}{\beta}\right). \tag{17}$$

The sth raw moment of X with the BIII-Pareto distribution using Mellin transform is

$$\mu_s' = \frac{\mathrm{M}\left\{f\left(x\right);\mathrm{s}+1\right\}}{\mathrm{M}\left\{f\left(x\right);1\right\}}.$$

The kth moment about the mean of X is determined from the relationship

$$\mu_k = E\left[X-E\left(X\right)\right]^k = \sum_{j=1}^{k}\binom{k}{j}\left(-1\right)^j \mu_j'\mu_{(k-j)}'.$$

The Pearson's measure of skewness γ_1, Kurtosis β_2, moment generating function and cumulants can be calculated from

$$\gamma_1 = \frac{\mu_3}{(\mu_2)^{\frac{3}{2}}},\ \beta_2 = \frac{\mu_4}{(\mu_2)^2}, M_X\left(t\right) = \sum_{r=1}^{\infty}\frac{t^r}{\mathrm{r!}}E\left(X\right)^r \ and\ k_r = E\left(X^r\right) - \sum_{c=1}^{r-1}\binom{r-1}{c-1}k_c E\left(X^{r-c}\right).$$

The numerical measures of the median, mean, variance, skewness and kurtosis of the BIII-Pareto distribution for selected values of the parameters to illustrate their effect on these measures.

Table 1: Median, mean, standard deviation, skewness and Kurtosis of the BIII-Pareto Distribution

Parameters $\alpha,\beta,\kappa,\theta$	Median	Mean	Standard Deviation	Skewness	Kurtosis
1,2,2,1	1.4136	1.5296	0.4989	16.5801	1914
1,2,3,1	1.2596	1.3169	0.2462	4.1497	87.5969
1,3,2,1	1.4138	1.4635	0.2586	3.4796	70.4517
1,3,3,1	1.2597	1.2852	0.1436	2.2267	20.2875
0.5,3,3,1	1.1917	1.2138	0.1372	2.0026	17.6597
1,3,3.5,1	1.2190	1.2393	0.1172	1.8895	12.0125
1,3.5,3,1	1.2599	1.2787	0.1194	1.7343	11.1507
1,3.5,3.5,1	1.2190	1.2340	0.0980	1.6220	9.7836
1,4,4,1	1.8909	1.1985	0.0713	1.4059	9.2767
0.1,5,5,1	1.0455	1.0587	0.0533	0.9902	3.9426
0.1,6,5,1	1.0562	1.0647	0.0514	0.6883	3.0007
0.4,7.8,6,1	1.1050	1.1039	0.0282	0.0059	3.7621
0.25,7,6,1	1.0830	1.0813	0.0323	0.0310	3.0416
0.25,7.24,5.5,1	1.0999	1.0978	0.0379	0.0077	3.0475
0.25,7.25,5.55,1	1.0990	11.0969	0.0375	0.0038	3.0434
0.1,6,6,1	1.0466	1.0535	0.0423	0.6710	2.9399
0.1,7,7,1	1.0462	1.0494	0.3482	0.4361	2.4382
0.1,8,8,1	1.0448	1.0459	0.0293	0.2644	2.2561
0.3,8,6,1	1.0986	1.0966	0.0305	-0.1046	3.3282
0.2,8,5,1	1.1054	1.1024	0.0423	-0.0584	2.7897
0.3,8,7,1	1.0840	1.0822	0.0258	-0.1243	3.2567
0.3,10,7,1	1.0878	1.0856	0.0220	-0.3383	3.3906

4.2 Moments of Order Statistics

Moments of order statistics have applications in reliability and life testing. Moments of order statistics are also designed for the replacement policy with the prediction of failure of the future items determined from few early failures.

The pdf of the m$^{\text{th}}$ order statistic $X_{\text{m}:n}$ for the BIII-Pareto distribution is

$$f\left(x_{\text{m}:n}\right)=\frac{\sum_{j=0}^{n-m}(-1)^j\binom{n-m}{j}}{B(\text{m},n-m+1)}\alpha\beta\frac{\kappa}{\theta}\left(\frac{x}{\theta}\right)^{\kappa-1}\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta-1}\left\{1+\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-(\alpha m+\alpha j+1)},x>\theta.$$

Moments about the origin of $X_{\text{m}:n}$ are given by

$$E\left(X^r{}_{\text{m}:n}\right)=\int_{\theta}^{\infty}x^r\frac{\sum_{j=0}^{n-m}(-1)^j\binom{n-m}{j}}{B(\text{m},n-m+1)}\alpha\beta\frac{\kappa}{\theta}\left(\frac{x}{\theta}\right)^{\kappa-1}\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta-1}\left\{1+\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-(\alpha m+\alpha j+1)}dx,$$

$$E\left(X^r{}_{\text{m}:n}\right)=\alpha\theta^r\frac{\sum_{j=0}^{n-m}(-1)^j\binom{n-m}{j}}{B(\text{m},n-m+1)}\sum_{i=0}^{\frac{r}{\kappa}}\binom{\frac{r}{\kappa}}{i}B\left(1-\frac{i}{\beta},\alpha(m+j)+\frac{i}{\beta}\right),\ \text{r}=1,2,3,.. \qquad (18)$$

Mean of $X_{\text{m}:n}$ is

$$E\left(X_{\text{m}:n}\right)=\alpha\theta\frac{\sum_{j=0}^{n-m}(-1)^j\binom{n-m}{j}}{B(\text{m},n-m+1)}\sum_{i=0}^{\frac{1}{\kappa}}\binom{\frac{1}{\kappa}}{i}B\left(1-\frac{i}{\beta},\alpha\left(m+j\right)+\frac{i}{\beta}\right). \tag{19}$$

4.3 Incomplete Moments

Incomplete moments are used in mean inactivity life, mean residual life function and other inequality measures.

The lower incomplete moments for random variable X with the BIII-Pareto distribution are

$$M_r'(z)=E_{X\le z}\left(X^r\right)=\int_\theta^z x^r\alpha\beta\frac{\kappa}{\theta}\left(\frac{x}{\theta}\right)^{\kappa-1}\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta-1}\left\{1+\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\alpha-1}dx,$$

Let $\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}=w,\quad x=\theta\left(w^{-\frac{1}{\beta}}+1\right)^{\frac{1}{\kappa}},\quad x^r=\theta^r\left(w^{-\frac{1}{\beta}}+1\right)^{\frac{r}{\kappa}}$, then

$$E_{X\le z}\left(X^r\right)=\alpha\theta^r\int_0^{\left[\left(\frac{z}{\theta}\right)^{\kappa}-1\right]^{-\beta}}\left(1+w^{-\frac{1}{\beta}}\right)^{\frac{r}{\kappa}}\{1+w\}^{-\alpha-1}dw,$$

$$M_r'(z)=E_{X\le z}\left(X^r\right)=\alpha\theta^r\sum_{i=0}^{\frac{r}{\kappa}}\binom{\frac{r}{\kappa}}{i}\left[\begin{array}{l}B\left(1-\frac{i}{\beta},\alpha+\frac{i}{\beta}\right)-\\ \qquad B\left(\left[\left(\frac{z}{\theta}\right)^{\kappa}-1\right]^{-\beta};1-\frac{i}{\beta},\alpha+\frac{i}{\beta}\right)\end{array}\right],\ \text{r}=1,2,3,\ldots, \tag{20}$$

where $B(\text{z}\ ;.,.)$ is the incomplete beta function.

The upper incomplete moments for the random variable X with the BIII-Pareto distribution are

$$E_{X\ge z}\left(X^r\right)=\int_z^\infty x^r\alpha\beta\frac{\kappa}{\theta}\left(\frac{x}{\theta}\right)^{\kappa-1}\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta-1}\left\{1+\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\alpha-1}dx,$$

$$E_{X\ge z}\left(X^r\right)=\alpha\theta^r\sum_{i=0}^{\frac{r}{\kappa}}\binom{\frac{r}{\kappa}}{i}\left[B\left(\left[\left(\frac{z}{\theta}\right)^{\kappa}-1\right]^{-\beta};\quad 1-\frac{i}{\beta},\alpha+\frac{i}{\beta}\right)\right]. \tag{21}$$

The mean deviation about mean is $MD_{\bar{X}}=E\left|X-\mu_1^1\right|=2\mu_1^1F\left(\mu_1^1\right)-2\mu_1^1M_1'\left(\mu_1^1\right)$ and the mean deviation about median is $MD_M=E\left|X-M\right|=2MF\left(M\right)-2MM_1'\left(M\right)$ where $\mu_1^/=E\left(X\right)$ and $M=Q_{\frac{1}{2}}$. Bonferroni and Lorenz curves for a specified probability p are computed by $\text{B}(p)=M_1'(q)/p\mu_1^1$ and $L(p)=M_1'(q)/\mu_1^1$, where $q=Q_p$.

4.4 Residual Life functions

The residual life, say $m_n(z)$, of X with the BIII-Pareto distribution is

$$m_n(z) = E\left[(X-z)^n \middle| X > z\right],$$

$$m_n(z) = \frac{1}{S(z)}\int_z^\infty (x-z)^s f(x)dx,$$

$$m_n(z) = \frac{1}{S(z)}\sum_{s=0}^{n}\binom{n}{s}(-z)^{n-s} E_{X>z}\left(X^s\right),$$

$$m_n(z) = \frac{1}{S(z)}\sum_{s=0}^{n}\binom{n}{s}(-z)^{n-s}\alpha\theta^s\sum_{i=0}^{\frac{s}{\kappa}}\binom{\frac{s}{\kappa}}{i}B\left\{\left[\left(\frac{z}{\theta}\right)^{\kappa}-1\right]^{-\beta};1-\frac{i}{\beta},\alpha+\frac{i}{\beta}\right\}. \qquad (22)$$

The average remaining lifetime of a component at time z, say $m_1(z)$, or life expectancy known as mean residual life (MRL) function, is given by

$$m_1(z) = \frac{1}{S(z)}\sum_{s=0}^{1}\binom{1}{s}(-z)^{1-s}\alpha\theta^s\sum_{i=0}^{\frac{s}{\kappa}}\binom{\frac{s}{\kappa}}{i}B\left(\left[\left(\frac{z}{\theta}\right)^{\kappa}-1\right]^{-\beta};\ 1-\frac{i}{\beta},\alpha+\frac{i}{\beta}\right). \qquad (23)$$

The reverse residual life, say $M_n(z)$, of X with the BIII-Pareto distribution is

$$M_n(z) = E\left[(z-X)^n \middle| X \le z\right],$$

$$M_n(z) = \frac{1}{F(z)}\int_a^z (z-x)^n f(x)dx,$$

$$M_n(z) = \frac{1}{F(z)}\sum_{s=0}^{n}(-1)^s\binom{n}{s}z^{n-s}E_{X\le z}\left(X^s\right),$$

$$M_n(z) = \frac{1}{F(z)}\sum_{s=0}^{n}(-1)^s\binom{n}{s}z^{n-s}\alpha\theta^s\sum_{i=0}^{\frac{s}{\kappa}}\binom{\frac{s}{\kappa}}{i}\left(\begin{array}{l} B\left(1-\frac{i}{\beta},\alpha+\frac{i}{\beta}\right)- \\ \qquad B\left\{\left[\left(\frac{z}{\theta}\right)^{\kappa}-1\right]^{-\beta};\ 1-\frac{i}{\beta},\alpha+\frac{i}{\beta}\right\}\end{array}\right). \qquad (24)$$

The waiting time z for the failure of a component has passed with condition that this failure had happened in the interval [0, z] is called mean waiting time (MWT) or mean inactivity time. The waiting time z for the failure of a component of X with BIII-Pareto distribution is defined by

$$M_1(z) = \frac{1}{F(z)}\sum_{s=0}^{1}(-1)^s\binom{1}{s}z^{1-s}\alpha\theta^s\sum_{i=0}^{\frac{s}{\kappa}}\binom{\frac{s}{\kappa}}{i}\left(\begin{array}{l} B\left(1-\frac{i}{\beta},\alpha+\frac{i}{\beta}\right)- \\ \qquad B\left\{\left[\left(\frac{z}{\theta}\right)^{\kappa}-1\right]^{-\beta};\ 1-\frac{i}{\beta},\alpha+\frac{i}{\beta}\right\}\end{array}\right). \qquad (25)$$

5. RELIABILITY MEASURES

In this section, reliability measures are studied.

5.1 Stress-strength Reliability for the BIII-Pareto Distribution

Le $X_1 \sim BIII-Pareto(\alpha_1,\beta,\kappa,\theta), X_2 \sim BIII-Pareto(\alpha_2,\beta,\kappa,\theta)$ such that X_1 represents strength and X_2 represents stress. Then, reliability of the component for the BIII-Pareto distribution is

$$R = \Pr(X_2 < X_1) = \int_{\theta}^{\infty} f_{x_1}(x) F_{x_2}(x)\,dx$$

$$R = \int_{\theta}^{\infty} \alpha_1 \beta \frac{\kappa}{\theta}\left(\frac{x}{\theta}\right)^{\kappa-1}\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta-1}\left\{1+\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\alpha_1-1}\left\{1+\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\alpha_2} dx$$

$$R = \int_{\theta}^{\infty} \alpha_1 \beta \frac{\kappa}{\theta}\left(\frac{x}{\theta}\right)^{\kappa-1}\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta-1}\left\{1+\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\alpha_1-\alpha_2-1} = \frac{\alpha_1}{(\alpha_1+\alpha_2)}. \qquad (26)$$

Therefore (i) R is independent of $\beta, \kappa \text{ and } \theta$ (ii) for $\alpha_1 = \alpha_2$, R=0.5, X_1 *and* X_2 are independently and identically distributed (i.i.d.) and there is equal chance that X_1 is bigger than X_2.

5.2 Estimation of Multicomponent Stress-Strength System Reliability for the BIII-Pareto Distribution

Suppose a machine has at least "s" components working out of "m" component. The strengths of all components of the system are $X_1, X_2, ... X_m$ and stress Y is applied to the system. Both strengths $X_1, X_2, ... X_m$ and stress Y are i.i.d.. The cdf of Y is G and F is cdf of X. The reliability of a machine is the probability that the machine functions properly i.e.

$$R_{s,m} = P(strengths > stress) = P[atleast\, "s" of\, (X_1, X_2, ... X_m)\, exceed\, Y]. \qquad (27)$$

Let $X \sim BIII-Pareto(\alpha_1,\beta,\theta,\kappa), Y \sim BIII-Pareto(\alpha_2,\beta,\theta,\kappa)$ with unknown shape parameters α_1 *and* α_2 and common scale parameter θ, where X and Y are independently distributed. The reliability in multicomponent stress- strength for the BIII-Pareto distribution is:

$$R_{s,m} = \sum_{l=s}^{m}\binom{m}{l}\int_{-\infty}^{\infty}[1-F(y)]^{l}[F(y)]^{m-l}\,dG(y)$$ (Bhattacharyya and Johnson; 1974), (28)

$$R_{s,m}=\sum_{\ell=s}^{m}\binom{m}{\ell}\int_{\theta}^{\infty}\left(1-\left\{1+\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\alpha_1}\right)^{\ell}\left(\left\{1+\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\alpha_1}\right)^{(m-\ell)}\alpha_2\beta\frac{\kappa}{\theta}\left(\frac{x}{\theta}\right)^{\kappa-1}\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta-1}\left\{1+\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\alpha_2-1}dy.$$

Letting $t=\left[1+\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right]^{-\alpha_2}$, we obtain $R_{s,m}=\sum_{\ell=s}^{m}\binom{m}{\ell}\int_{0}^{1}\left(1-t^{\nu}\right)^{\ell}t^{\nu(m-\ell)}dt.$

Let $z=t^{\nu}, t=z^{\frac{1}{\nu}},\ dt=\frac{1}{\nu}z^{\frac{1}{\nu}-1}dz$, then $R_{s,m}=\sum_{\ell=s}^{m}\binom{m}{\ell}\int_{0}^{1}\left(1-t^{\nu}\right)^{\ell}t^{\nu(m-\ell)}dt,$

$$R_{s,m}=\sum_{\ell=s}^{m}\binom{m}{\ell}\int_{0}^{1}(1-z)^{\ell}z^{(m-\ell)}\frac{1}{\nu}z^{\frac{1}{\nu}-1}dz,$$

$$R_{s,m}=\frac{1}{\nu}\sum_{\ell=s}^{m}\binom{m}{\ell}B\left(\ell+1,m-\ell+\frac{1}{\nu}\right).$$

$$R_{s,m}=\frac{1}{\nu}\sum_{\ell=s}^{m}\frac{m!}{(m-\ell)!}\left[\prod_{j=0}^{\ell}\left(m-j+\frac{1}{\nu}\right)\right]^{-1}\ \textit{where}\ \nu=\frac{\alpha_1}{\alpha_2}. \qquad (29)$$

The probability $R_{s,m}$ in (29) is called multicomponent stress-strength model reliability.

6. CHARACTERIZATIONS

In this section, the BIII-Pareto distribution is characterized through: (i) conditional expectation; (ii) ratio of truncated moments (iii) elasticity function (iv) reverse hazard rate function (v) Characterization based on the conditional expectation of certain function of the random variable (vi) conditional expectation of lower record values and (vii) conditional expectation of lower record values with spacing.

6.1 Characterization via Conditional Expectation

Here the BIII-Pareto distribution is characterized via conditional expectation.

Proposition 6.1.1: Let $X:\Omega\to(\theta,\infty)$ be a continuous random variable with cdf $F(x)$ ($0<F(x)<1$ *for* $x\geq\theta$), then for $\alpha>1$, X has pdf (4) if and only if

$$E\left[\left[\left(\frac{X}{\theta}\right)^{\kappa}-1\right]^{-\beta}\middle|X<t\right]=\frac{1}{(\alpha-1)}\left[1+\alpha\left[\left(\frac{t}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right]\quad \textit{for}\ t>\theta. \qquad (30)$$

Proof. If X has pdf (4), then

$$E\left[\left[\left(\frac{X}{\theta}\right)^{\kappa}-1\right]^{-\beta}\middle|X<t\right]=\left(F(t)\right)^{-1}\int_{a}^{t}\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}f(x)dx$$

$$=\left(F(t)\right)^{-1}\int_{a}^{t}\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\times$$

$$\alpha\beta\frac{\kappa}{\theta}\left(\frac{x}{\theta}\right)^{\kappa-1}\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta-1}\left\{1+\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-\alpha-1}dx.$$

Upon integration by parts and simplification, we obtain

$$E\left[\left[\left(\frac{X}{\theta}\right)^{\kappa}-1\right]^{-\beta}\middle| X<t\right]=\frac{1}{(\alpha-1)}\left[1+\alpha\left[\left(\frac{t}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right],\ for\ t>\theta.$$

Conversely, if proposition 6.1.1 holds, then

$$\int_{\theta}^{t}\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}f(x)dx=\frac{F(t)}{(\alpha-1)}\left[1+\alpha\left[\left(\frac{t}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right]. \qquad (31)$$

Differentiating (31) with respect to t, we obtain

$$\left[\left(\frac{t}{\theta}\right)^{\kappa}-1\right]^{-\beta}f(t)=\frac{f(t)}{(\alpha-1)}\left[1+\alpha\left[\left(\frac{t}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right]-\frac{F(t)}{(\alpha-1)}\left[\alpha\beta\frac{\kappa}{\theta}\left(\frac{x}{\theta}\right)^{\kappa-1}\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta-1}\right].$$

After simplification and integration, we arrive at $F(t)=\left[1+\left[\left(\frac{t}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right]^{-\alpha}, t\geq\theta.$

6.2 Characterization of the BIII-Pareto Distribution through Ratio of Truncated Moments

The BIII-Pareto distribution is characterized using Theorem 1 (Glänzel; 1987) on the basis of a simple relationship between two truncated moments of X. Theorem 1 is given in Appendix A.

Proposition 6.2: Let $X:\Omega\to(\theta,\infty)$ be a continuous random variable. Let $h_1(x)=\frac{1}{\alpha}\left\{1+\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{\alpha+1}, x>\theta$ and $h_2(x)=2\alpha^{-1}\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\left\{1+\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{\alpha+1}, x>\theta$.

The random variable X has pdf (4) if and only if, the function $p(x)$ has the form $p(x)=\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{\beta}, x>\theta.$

Proof. If X has pdf (4), then

$$\left(1-F(x)\right)E\left(h_1(x)\middle| X\geq x\right)=\left[\left(\frac{x}{\Theta}\right)^{\kappa}-1\right]^{-\beta}, x>\Theta,$$

$$\left(1-F(x)\right)E\left(h_2(x)\middle| X\geq x\right)=\left[\left(\frac{x}{\Theta}\right)^{\kappa}-1\right]^{-2\beta}, x>\Theta,$$

$$\frac{E[h_1(x)|X \geq x]}{E[h_2(x)|X \geq x]} = p(x) = \left[\left(\frac{x}{\theta}\right)^{\kappa} - 1\right]^{\beta}, x > \theta.$$

Conversely, if $p(x)$ has the given form, then $p'(x) = \beta \frac{\kappa}{\theta}\left(\frac{x}{\theta}\right)^{\kappa-1}\left[\left(\frac{x}{\theta}\right)^{\kappa} - 1\right]^{\beta-1}$.

The differential equation $s'(x) = \frac{p'(x)h_2(x)}{p(x)h_2(x) - h_1(x)} = 2\beta \frac{\kappa}{\theta}\left(\frac{x}{\theta}\right)^{\kappa-1}\left[\left(\frac{x}{\theta}\right)^{\kappa} - 1\right]^{-1}$ has solution

$s(x) = \ln\left[\left(\frac{x}{\theta}\right)^{\kappa} - 1\right]^{2\beta}, x > \theta.$

Therefore, in light of theorem 1, X has pdf (4)

Corollary 6.2.1: Let $X : \Omega \to (\theta, \infty)$ be a continuous random variable and let $h_2(x) = 2\alpha^{-1}\left[\left(\frac{x}{\theta}\right)^{\kappa} - 1\right]^{-\beta}\left\{1 + \left[\left(\frac{x}{\theta}\right)^{\kappa} - 1\right]^{-\beta}\right\}^{\alpha+1}$, $x > \theta$. The pdf of X is (4) if and only if there exist functions $p(x)$ *and* $h_1(x)$ (defined in Theorem 1), satisfying the differential equation

$$\frac{p'(x)}{p(x)h_2(x) - h_1(x)} = \alpha\beta \frac{\kappa}{\theta}\left(\frac{x}{\theta}\right)^{\kappa-1}\left[\left(\frac{x}{\theta}\right)^{\kappa} - 1\right]^{\beta-1}\left\{1 + \left[\left(\frac{x}{\theta}\right)^{\kappa} - 1\right]^{-\beta}\right\}^{-\alpha-1}. \qquad (32)$$

Remark 6.2.1: The general solution of (32) is

$$p(x) = \left[\left(\frac{x}{\theta}\right)^{\kappa} - 1\right]^{2\beta}\left[\int\left(-\alpha\beta \frac{\kappa}{\theta}\left(\frac{x}{\theta}\right)^{\kappa-1}\left[\left(\frac{x}{\theta}\right)^{\kappa} - 1\right]^{-\beta-1}\left\{1 + \left[\left(\frac{x}{\theta}\right)^{\kappa} - 1\right]^{-\beta}\right\}^{-\alpha-1} h_1(x)\right)dx + D\right],$$

where D is a constant.

6.3 Characterization of the BIII-Pareto Distribution Based on Reverse Hazard Rate Function

In this sub-section, the BIII-Pareto distribution is characterized via reverse hazard rate function.

Definition 6.3.1: The reverse hazard function $r_F(x)$, of a twice differentiable function F, satisfies the differential equation

$$\frac{d}{dx}\left[\ln f(x)\right] = \frac{r_F'(x)}{r_F(x)} + r_F(x).$$

Proposition 6.3.1: Let $X : \Omega \to (\theta, \infty)$ be continuous random variable. The pdf of X is (4) if and only if its reverse hazard rate function, $r_F(x)$ satisfies the first order differential equation

$$r_F'(x) - \frac{(\kappa-1) r_F(x)}{x} = \alpha\beta\kappa\theta^{-\kappa} x^{\kappa-1} \frac{d}{dx}\left[\frac{\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta-1}}{1+\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{\beta}}\right]. \qquad (33)$$

Proof If X has pdf (4), then (33) surely holds. Now if (33) holds, then

$$\frac{d}{dx}\left[r_F(x)\, x^{-\kappa+1}\right] = \alpha\beta\kappa\theta^{-\kappa} \frac{d}{dx}\left[\frac{\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta-1}}{1+\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{\beta}}\right],$$

or $$r_F(x) = \alpha\beta\frac{\kappa}{\theta}\left(\frac{x}{\theta}\right)^{\kappa-1}\left\{\frac{\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta-1}}{1+\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{\beta}}\right\}, \quad x>\theta,$$

which is the reverse hazard rate function of the BIII-Pareto distribution.

6.4 Characterization via Elasticity Function

In this sub-section, the BIII-Pareto distribution is characterized via elasticity.

Definition 6.4.1: Let $X:\Omega\to(\theta,\infty)$ be a continuous random variable with pdf $f(x)$.The elasticity function $e_F(x)$ is a twice differentiable function satisfying the differential equation

$$\frac{d}{dx}\left[\ln f(x)\right] = \frac{e_F'(x)}{e_F(x)} + \frac{e_F(x)}{x} - \frac{1}{x}.$$

Proposition 6.4.1: Let $X:\Omega\to(\theta,\infty)$ be continuous random variable. The pdf of X is (4) if its elasticity function, $e_F(x)$, satisfies the first order differential equation

$$\left\{\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]+\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{\beta+1}\right\}e_F'(x)+\frac{\kappa}{\theta}\left(\frac{x}{\theta}\right)^{\kappa-1}\left\{(\beta+1)\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{\beta}+1\right\}e_F(x)=\alpha\beta\kappa\frac{\kappa}{\theta}\left(\frac{x}{\theta}\right)^{\kappa-1}. \qquad (34)$$

Proof If X has pdf (4), then (34) surely holds. Now if (34) holds, then

$$\frac{d}{dx}\left[\left\{\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{\beta+1}+\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]\right\}e_F(x)\right] = \alpha\beta\kappa\frac{d}{dx}\left[\left(\frac{x}{\theta}\right)^{\kappa}\right],$$

or $$e_F(x) = \alpha\beta\kappa\left(\frac{x}{\theta}\right)^{\kappa}\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta-1}\left\{1+\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-1}, \; x>\theta,$$

which is the elasticity function of the BIII-Pareto distribution.

6.5 Characterization based on the conditional expectation of certain function of the random variable

In this subsection we employ a single function ψ of X and characterize the distribution of X in terms of the truncated moment of $\psi(X)$. The following proposition has already appeared in Hamedani's previous work (2013), so we will just state it here which can be used to characterize the BIII-Pareto distribution.

Proposition 6.5.1. Let $X:\Omega \to (e,f)$ be a continuous random variable with cdf F. Let $\psi(x)$ be a differentiable function on (e, f) with $\lim_{x\to f^-}\psi(x)=1$. Then for $\delta \neq 1$,

$$E[\psi(X)\,|\,X \le x] = \delta\psi(x),\ x \in (e,f) \text{ implies that } \psi(x) = (F(x))^{\frac{1}{\delta}-1},\ x \in (e,f).$$

Remark 6. 5.1. For $(e,f) = (\theta,\infty)$, $\psi(x) = \left\{1+\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right\}^{-1}$ *and* $\delta = \frac{\alpha}{\alpha+1}$.

Proposition 6.5.1 provides a characterization of the BIII-Pareto distribution.

6.6 Characterization via Conditional Expectation of Record Values

Faizan and Khan (2011) and Khan and Faizan (2014) characterized distributions via conditional expectation of lower record values. We characterize BIII-Pareto distribution via conditional expectation of the lower record values.

Proposition 6.6.1: Let $X:\Omega \to (a,b)$ be a continuous random variable with cdf $F(x)$ and pdf $f(x)$. Let $X_{L(r)}$ be the *rth* record value of a random sample $X_1, X_2, \ldots X_n$ .Then, for two successive values $X_{L(\ell)}$ and $X_{L(s)}$, $1 \le m < r < s \le n$,

$$\sum_{i=r}^{m} c_i E[(h\,X_{L(\ell)} \,|\, X_{L(r)} = x] = \frac{1}{a}\sum_{i=r}^{m} \ell c_i \text{ with } \ell = i-1,\ i,$$

if and only if $F(x) = e^{-ah(x)}$ $a > 0$, where c_i are real numbers satisfying $\sum_{i=r}^{m} c_i = 0$ with $c_i \neq 0$ and $h(x)$ is differentiable function of *x*.

Remark 6.6.1: Taking $a = \alpha$, $h(x) = \ln\left[1+\left[\left(\frac{x}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right]$, Proposition 6.6.1 provides a characterization of the BIII-Pareto distribution.

6.7 Characterizations Based on the Conditional Expectation of the Lower Record Values with Spacing

We characterize the BIII-Pareto distribution via conditional expectation of the lower record values with spacing.

Proposition 6.7.1: Let $X:\Omega \to (a_1,a_2)$ be a continuous variable with pdf f(x) and cdf F(x), where a_1 and a_2 may be finite or infinite. Let $X_{L(r)}$ be the rth lower record value, then the conditional expectation of $X_{L(\kappa)}$ given $X_{L(\ell)}$, for $1 \le \ell < \kappa$, is

$$E\left[X_{L(\kappa)} \middle| X_{L(\ell)}\right] = a_{\kappa|\ell}\,h(x) + b_{\kappa|\ell},\ \ell = r, r+1.$$

If and only if $F(a)=\left[a\,h(x)+b\right]^{c}$, $a_1<x<a_2$ and h(x) is a monotonic and differential function of x such that $h(x)\to 0$ as $x\to a_2$ and $F(x)\to 0$ as $x\to a_1$ where $a_{\kappa|\ell}=\left(1+c^{-1}\right)^{\ell-\kappa}$ *and* $b_{\kappa|\ell}=-\frac{b}{a}\left(1-a_{\kappa|\ell}\right)$.

Remark 6.7.1: Taking $a=1,\ b=1,\ h(x)=\left[\left(\frac{t}{\theta}\right)^{\kappa}-1\right]^{-\beta}$, $c=-\alpha$, Proposition 6.7.1 provides a characterization of the BIII-Pareto distribution.

7. MAXIMUM LIKELIHOOD ESTIMATION

In this section, parameter estimates are derived using maximum likelihood method. The log likelihood function for the BIII-Pareto distribution with the vector of parameters $\Phi=(\alpha,\beta,\theta,\kappa)$ is

$$\ell=\ell(\Phi)=\left\{\begin{array}{l} n\ln\alpha+n\ln\beta+n\ln\kappa-n\kappa\ln\theta+(\kappa-1)\ln x_i- \\ \qquad (\beta+1)\sum_{i=1}^{n}\ln\left[\left(\frac{x_i}{\theta}\right)^{\kappa}-1\right]-(\alpha+1)\sum_{i=1}^{n}\ln\left[1+\left[\left(\frac{x_i}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right]\end{array}\right\}, \tag{35}$$

where θ is assumed to be known. In order to compute the estimates of the parameters of the BIII-P distribution, the following nonlinear equations must be solved simultaneously:

$$\frac{\partial\ell}{\partial\alpha}=\frac{n}{\alpha}-\sum_{i=1}^{n}\ln\left[1+\left[\left(\frac{x_i}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right]=0, \tag{36}$$

$$\frac{\partial\ell}{\partial\beta}=\left\{\begin{array}{l}\frac{n}{\beta}-\sum_{i=1}^{n}\ln\left[\left(\frac{x_i}{\theta}\right)^{\kappa}-1\right]+ \\ \qquad (\alpha+1)\sum_{i=1}^{n}\left[1+\left[\left(\frac{x_i}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right]^{-1}\left[\left(\frac{x_i}{\theta}\right)^{\kappa}-1\right]^{-\beta}\ln\left[\left(\frac{x_i}{\theta}\right)^{\kappa}-1\right]\end{array}\right\}=0, \tag{37}$$

$$\frac{\partial\ell}{\partial\kappa}=\left\{\begin{array}{l}\frac{n}{\kappa}-n\ln\theta+\ln x_i-(\beta+1)\sum_{i=1}^{n}\left[\left(\frac{x_i}{\theta}\right)^{\kappa}-1\right]^{-1}\left(\frac{x_i}{\theta}\right)^{\kappa}\ln\left(\frac{x_i}{\theta}\right)+ \\ \qquad (\alpha+1)\beta\sum_{i=1}^{n}\left(\frac{x_i}{\theta}\right)^{\kappa}\ln\left(\frac{x_i}{\theta}\right)\left[\left(\frac{x_i}{\theta}\right)^{\kappa}-1\right]^{-\beta-1}\left[1+\left[\left(\frac{x_i}{\theta}\right)^{\kappa}-1\right]^{-\beta}\right]^{-1}\end{array}\right\}=0. \tag{38}$$

8. SIMULATION STUDY

In this Section, we perform a simulation study by using selected the BIII-Pareto distributions. To see the performance of MLE's of these distributions, we generate 1,000 samples of sizes 50, 100, 200 and 400 with its quantile function of the BIII-Pareto distribution. All results related to MLEs have been obtained by the optim-CG routine in the R programme. The results of the simulations are reported in Table 2. From this Table, we observe that the estimates approach true values as the sample size increases whereas the standard deviations of the estimates decrease, as expected.

Table 2. Empirical means and standard deviations (in parenthesis) for selected the BIII-Pareto distributions

Sample sizes	True Parameters $\alpha,\beta,\theta,\kappa$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\kappa}$
n=50	5,2,0.5,3	4.9994 (0.0123)	2.0248 (0.1470)	0.5023 (0.0116)	3.0184 (0.0891)
	3,2,1,5	2.9977 (0.0923)	2.0456 (0.1777)	1.0041 (0.0098)	5.0531 (0.2205)
	0.7,6,10,0.5	0.7611 (0.2549)	6.0368 (0.5902)	10.0343 (0.2120)	0.5046 (0.0276)
	10,5,5,0.8	10.0117 (0.0809)	5.0438 (0.3371)	5.0650 (0.3111)	0.8063 (0.0374)
	7,0.8,0.3,10	6.9729 (0.2484)	0.8122 (0.0744)	0.3009 (0.0055)	9.9854 (0.1679)
n=100	5,2,0.5,3	4.9998 (0.0089)	2.0142 (0.1037)	0.5016 (0.0085)	3.0082 (0.0703)
	3,2,1,5	2.9915 (0.0532)	2.0099 (0.1652)	0.9997 (0.0097)	5.0338 (0.2110)
	0.7,6,10,0.5	0.7525 (0.1315)	5.9842 (0.2278)	10.0081 (0.0717)	0.5036 (0.0194)
	10,5,5,0.8	9.9992 (0.0372)	5.0058 (0.2006)	5.0500 (0.1892)	0.8056 (0.0239)
	7,0.8,0.3,10	6.9977 (0.0171)	0.8104 (0.0555)	0.3006 (0.0042)	9.9998 (0.0117)
n=200	5,2,0.5,3	5.0003 (0.0084)	2.0021 (0.0835)	0.4996 (0.0067)	3.0006 (0.0038)
	3,2,1,5	2.9997 (0.0117)	1.9995 (0.0890)	1.0004 (0.0054)	5.0027 (0.1623)
	0.7,6,10,0.5	0.7134 (0.0862)	6.0069 (0.1953)	10.0078 (0.0586)	0.5019 (0.0137)
	10,5,5,0.8	10.0010 (0.0301)	5.0026 (0.1307)	5.0359 (0.1269)	0.8044 (0.0167)
	7,0.8,0.3,10	6.9994 (0.0042)	0.8042 (0.0330)	0.3002 (0.0028)	10.0002 (0.0032)
n=400	5,2,0.5,3	5.0001 (0.0065)	1.9943 (0.0527)	0.4999 (0.0038)	2.9994 (0.0382)
	3,2,1,5	2.9998 (0.0250)	1.9997 (0.0792)	0.9998 (0.0047)	5.0123 (0.1399)
	0.7,6,10,0.5	0.7093 (0.0624)	6.0022 (0.1539)	10.0027 (0.0511)	0.5011 (0.0092)
	10,5,5,0.8	9.9998 (0.0182)	4.9967 (0.1169)	5.0008 (0.1201)	0.8010 (0.0146)
	7,0.8,0.3,10	6.9999 (0.0006)	0.8032 (0.0282)	0.2999 (0.0020)	10.0002 (0.0019)

9. APPLICATIONS

The BIII-Pareto distribution is compared with IL-Pareto, LL-Pareto, Pareto and Burr III (BIII) distributions. Different goodness fit measures such as Cramer-von Mises (W), Anderson Darling (A), Kolmogorov- Smirnov statistics with p-values, Akaike information criterion (AIC), consistent Akaike information criterion (CAIC), Bayesian information criterion (BIC), Hannan-Quinn information criterion (HQIC) and likelihood ratio statistics ($-2\hat{\ell}$) are computed for monthly remission times of 128 cancer patients (Lee and Wang, 2003), average annual percent change in private health insurance premiums and strength of carbon fibers using R-Package.

The better fit corresponds to smaller W, A, K-S, AIC, CAIC, BIC, HQIC and $-2\hat{\ell}$ value. The maximum likelihood estimates (MLEs) of unknown parameters and values of goodness of fit measures are computed for BIII-Pareto distribution and its sub and competing models. The MLEs**,** their standard errors (in parentheses) and goodness-of-fit statistics like W, A, K-S (p-value) are given in table 3, 5 and 7. Table 4, 6 and 8 displays goodness-of-fit values.

9.1 Application I: Monthly Remission Times

Monthly remission of 128 cancer patients (bladder) are : "0.08,2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26,3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54,3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77,32.15, 2.64, 3.88,5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34,7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23,5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26,2.83, 4.33, 5.49,7.66,11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79,18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63 and 22.69".

Table 3: MLEs, their standard errors (in parentheses) and Goodness-of-fit statistics for cancer patients' data

Model	α	β	κ	θ	W	A	K-S p-value
BIII-Pareto	0.3813 (0.1034)	8.9283 (1.5955)	0.1382 (0.0051)	0.08	**0.0203**	**0.1425**	**0.038 (0.9929)**
IL-Pareto	23.9531 (4.7842)	---	0.8514 (0.0546)	0.08	0.5866	3.6424	0.1234 (0.04171)
LL- Pareto	---	5.2584 (0.3965)	0.1610 (0.0034)	0.08	0.1382	0.9346	0.053 (0.8675)
Pareto	---	---	0.2319 (0.0206)	0.08	0.2904	1.8785	0.4246 (<2.2e-16)
BIII	4.207 (0.4054)	1.0333 (0.0604)	---	---	0.3856	2.4543	0.1017 (0.1413)

Table 4: Goodness-of-fit statistics for cancer patients' data

Model	AIC	CAIC	BIC	HQIC	$-2\hat{\ell}$
BIII-Pareto	**818.9111**	**819.1062**	**827.4436**	**822.3777**	**812.9110**
IL-Pareto	868.1794	868.2762	873.8678	870.4905	864.1794
LL- Pareto	828.7483	828.8451	834.4367	831.0595	824.7484
Pareto	1081.182	1081.214	1084.026	1082.338	1079.182
BIII	857.3729	857.4689	863.0769	859.6905	853.3728

We can perceive that the BIII-Pareto distribution is best fitted model because the values of all criteria of goodness of fit are significantly smaller for BIII-Pareto distribution.

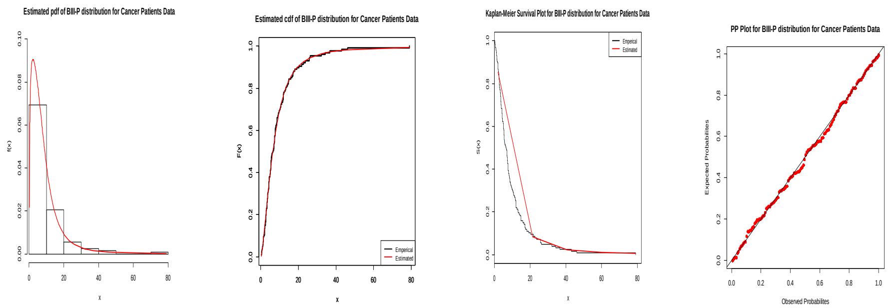

Fig. 5: Fitted pdf, cdf, survival and pp plots of the BIII-Pareto distribution for Cancer Patients Data

We can perceive that the BIII-Pareto distribution is best fitted to empirical data (Fig. 5).

9.2 Application II: Insurance Premiums

The average annual percent change in private health insurance premiums are: 14.4, 14.0, 15.4, 9.4, 11.7, 15.0, 24.9, 20.7, 12.5, 14.9, 12.6, 16.7, 13.8, 11.0, 12.9, 10.1, 1.9, 8.5, 16.5, 15.3, 13.3, 9.8, 8.4, 7.9, 3.7, 5.1, 4.6, 4.4, 5.4, 6.1, 8.0, 10.0, 11.2, 10.1, 6.4, 6.7, 5.7, 5.8.

Table 5: MLEs, their standard errors (in parentheses) and Goodness-of-fit statistics for insurance premiums

Model	α	β	κ	θ	W	A	K-S p-value
BIII-Pareto	0.2029 (0.1012)	11.9926 (4.7618)	0.3257 (0.0162)	1.9	**0.0272**	**0.2101**	**0.0783 (0.9772)**
IL-Pareto	20.0432 (7.7497)	----	2.151 (0.2811)	1.9	0.1975	1.1091	0.1455 (0.4136)
LL- Pareto	----	4.1537 (0.5758)	0.4227 (0.0211)	1.9	0.1392	0.8027	0.1002 (0.8516)
Pareto	----	----	0.6067 (0.0998)	1.9	0.1404	0.7914	0.3723 (7.021e-05)
BIII	29.0536 (9.3745)	1.6935 (0.1743)	----	----	0.2188	1.3453	0.1653 (0.2499)

Table 6: Goodness-of-fit statistics for insurance premiums

Model	AIC	CAIC	BIC	HQIC	$-2\hat{\ell}$
BIII-Pareto	**220.2731**	**221.0003**	**225.1058**	**221.9768**	**214.2730**
IL-Pareto	227.4998	227.8527	230.7216	228.6356	223.4998
LL- Pareto	224.5074	224.8603	227.7292	225.6432	220.5074
Pareto	282.3576	282.4719	283.9685	282.9255	280.3576
BIII	246.2924	246.6352	249.5675	247.4577	242.2924

We can perceive that the BIII-Pareto distribution is best fitted model because the values of all criteria of goodness of fit are significantly smaller for the BIII-Pareto distribution.

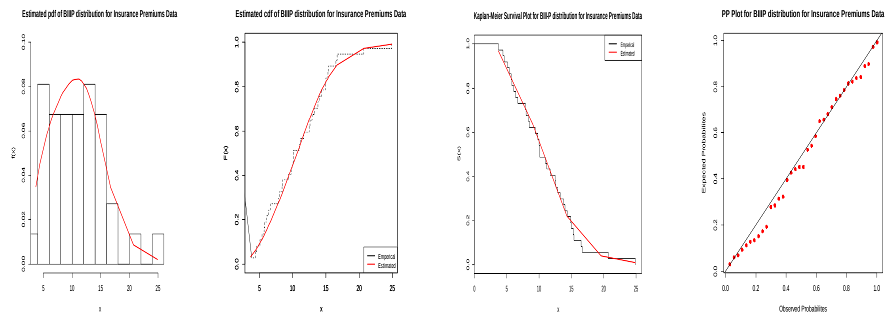

Fig. 6: Fitted pdf, cdf, survival and pp plots of the BIII-Pareto distribution for insurance premiums

We can perceive that the BIII-Pareto distribution is best fitted to empirical data (Fig.6)

9.3 Application III: Strength of Carbon Fibers

The tensile strength of 100 carbon fibers are: 3.7, 3.11, 4.42, 3.28, 3.75, 2.96, 3.39, 3.31, 3.15, 2.81, 1.41, 2.76, 3.19, 1.59, 2.17, 3.51, 0.84, 1.61, 1.57, 1.89, 2.74, 3.27, 2.41, 3.09, 2.43, 2.53, 2.81, 3.31, 2.35, 2.77, 2.68, 4.91, 1.57, 2.00,1.17, 2.17, 0.39, 2.79, 1.08, 2.88, 2.73, 2.87, 3.19, 1.87, 2.95, 2.67, 4.20, 2.85, 2.55, 2.17, 2.97, 3.68, 0.81, 1.22, 5.08, 1.69, 3.68, 4.70, 2.03, 2.82, 2.50, 1.47, 3.22, 3.15, 2.97, 2.93, 3.33, 2.56, 2.59, 2.83, 1.36, 1.84, 5.56, 1.12, 2.48, 1.25, 2.48, 2.03, 1.61, 2.05, 3.60, 3.11, 1.69, 4.90, 3.39, 3.22, 2.55, 3.56, 2.38, 1.92, 0.98, 1.59, 1.73, 1.71, 1.18, 4.38, 0.85, 1.80, 2.12, 3.65.

Table 7: MLEs, their standard errors (in parentheses) and Goodness-of-fit statistics for strength of carbon fibers

Model	α	β	κ	θ	W	A	K-S p-value
BIII-Pareto	0.2391 (0.0648)	13.1598 (2.6949)	0.3128 (0.008)	0.39	0.0517	**0.3213**	**0.0557 (0.9185)**
IL-Pareto	30.8544 (7.6556)	——	2.1562 (0.1611)	0.39	0.7029	3.9721	0.1591 (0.01334)
LL- Pareto	——	5.2540 (0.4515)	0.3770 (0.009)	0.39	0.4046	2.1981	0.1072 (0.2057)
Pareto	——	——	0.5476 (0.055)	0.39	0.4740	2.6314	0.4022 (2.442e-14)
BIII	4.9677 (0.5619)	2.2867 (0.148)	——	---	0.6239	3.4468	0.1476 (0.02556)

Table 8: Goodness-of-fit statistics for strength of carbon fibers

Model	AIC	CAIC	BIC	HQIC	$-2\widehat{\ell}$
BIII-Pareto	**279.1095**	**279.3621**	**286.8949**	**282.2595**	**273.1094**
IL-Pareto	319.259	319.384	324.4492	321.359	315.2590
LL- Pareto	297.4927	297.617	302.683	299.5927	293.4928
Pareto	494.3998	494.4411	496.995	495.4498	492.3998
BIII	327.7371	327.8608	332.9474	329.8458	323.7370

We can perceive that the BIII-Pareto distribution is best fitted model because the values of all criteria of goodness of fit are significantly smaller for the BIII-Pareto distribution.

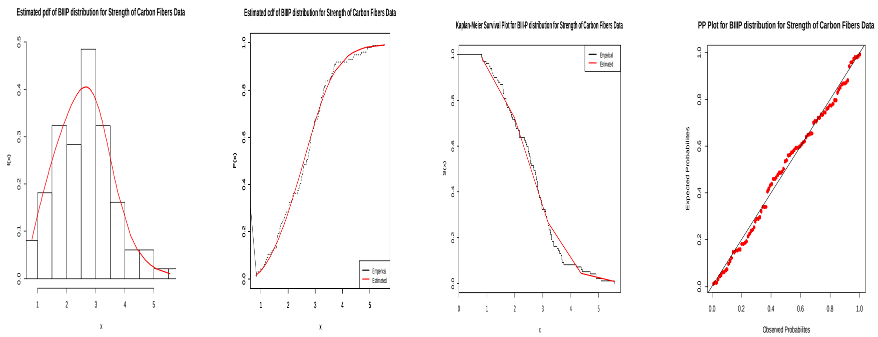

Fig. 7: Fitted pdf, cdf, survival and pp plots of the BIII-Pareto distribution for strength of carbon fibers

We can perceive that the BIII-Pareto distribution is best fitted to empirical data (Fig. 7).

10. CONCLUDING REMARKS

We have developed a flexible BIII-Pareto distribution on the basis of the T-X family technique. We have studied certain properties of this distribution including descriptive measures, sub-models, moments, factorial moments, incomplete moments, inequality measures, residual life functions, reliability measures and compounding. The BIII-Pareto distribution is characterized via different techniques. The MLEs for the BIII-Pareto distribution have been computed. The simulation study for the performance of the MLEs of the BIII-Pareto distribution with respect to sample size n is carried out. Applications of the BIII-Pareto model to real data sets (monthly remission times of cancer patients, health insurance premiums and strength of carbon fibers) are presented to show the significance and flexibility of the BIII-Pareto distribution. Goodness of fit shows that the BIII-Pareto distribution is the best fitted model. We have shown that the BIII-Pareto distribution is empirically better for lifetime applications.

ACKNOWLEDGMENT

No funding available for Project.

REFERENCES

1. Akinsete, A., Famoye, F., & Lee, C. (2008). The beta-Pareto distribution. *Statistics*, *42*(6), 547-563.
2. Alzaatreh, A., Famoye, F., & Lee, C. (2012). Gamma-Pareto distribution and its applications. *Journal of Modern Applied Statistical Methods*, *11*(1), 7
3. Alzaatreh, A., Famoye, F., & Lee, C. (2013). Weibull-Pareto distribution and its applications. *Communications in Statistics-Theory and Methods*, *42*(9), 1673-1691.
4. Alzaatreh, A., Lee, C., & Famoye, F. (2013). A new method for generating families of continuous distributions. *Metron*, *71*(1), 63-79.

5. Arnold, B. C. (2015). *Pareto distribution*. John Wiley & Sons, Ltd.
6. Bourguignon, M., Saulo, H., & Fernandez, R. N. (2016). A new Pareto-type distribution with applications in reliability and income data. *Physica A: Statistical Mechanics and its Applications*, *457*, 166-175
7. Choulakian, V., & Stephens, M. A. (2001). Goodness-of-fit tests for the generalized Pareto distribution. *Technometrics*, *43*(4), 478-484.
8. Dunning, K.A. and Hanson, J.N. (1977). Generalized Pearson distributions and nonlinear programing. *Journal of Statistical Computation and Simulation*, 6(2), 115-128.
9. Faizan, M. and Khan, M. I., (2011). A characterization of continuous distributions through lower record statistics. *ProabStat Forum*, 4, 39-43.
10. Faizan, M., Haque, Z., & Ansari, M. A. (2017). Characterizations of Distributions by Expected Values of Lower Record Statistics withSpacing. Journal of Modern Applied Statistical Methods, 16(2), 310-321. doi: 10.22237/jmasm/1509495360.
11. Farshchian, M., & Posner, F. L. (2010, May). The Pareto distribution for low grazing angle and high resolution X-band sea clutter. In *Radar Conference, 2010 IEEE* (pp. 789-793). IEEE.
12. Glänzel, W. A. (1987). *Characterization theorem based on truncated moments and its application to some distribution families, Mathematical Statistics and Probability Theory* (Bad Tatzmannsdorf, 1986), Vol. B, Reidel, Dordrecht, 75-84.
13. Hamedani, G.G.,(2013), On certain generalized gamma convolution distributions II, Technical Report No. 484, MSCS, Marquette University.
14. Isah, A., & Bala, S. S. (2017). The Generalized Weibull Pareto Distribution; Its Properties And Application. *ATBU Journal of Science, Technology and Education*, *5*(2), 139-154.
15. Khan, M. I., and Faizan, M. (2014). Characterization of distribution by conditional expectation of lower record values. *American Journal of Applied Mathematics and Statistics*, *2*(1), 7-9
16. Mahmoudi, E. (2011). The beta generalized Pareto distribution with application to lifetime data. *Mathematics and Computers in Simulation* 81:2414–2430.
17. Nadeau, T. P., & Teorey, T. J. (2003). A Pareto model for OLAP view size estimation. *Information Systems Frontiers*, *5*(2), 137-147.
18. Nasiru, S., & Luguterah, A. (2015). The new weibull-pareto distribution. *Pakistan Journal of Statistics and Operation Research*, *11*(1).
19. Nassar, M.M., Nada, N.K. (2011). The beta generalized Pareto distribution. *Journal of Statistics: Advances in Theory and Applications* 6:1–17.
20. Pareto, V. (1896). La courbe de la rpartition de la richesse. In *Recueil publi par la Facult de Droit l.occasion de l.exposition nationale suisse* (Eds) C. ViretGenton, Lausanne: Universit de Lausanne, pp. 373–387.
21. Pickands, J. (1975). Statistical inference using extreme order statistics. *Annals of Statistics* 3:119– 131.
22. Pisarenko, V. F., & Sornette, D. (2003). Characterization of the frequency of extreme earthquake events by the generalized Pareto distribution. *pure and applied geophysics*, *160*(12), 2343-2364.
23. Tahir, M. H., Cordeiro, G. M., Alzaatreh, A., Mansoor, M., & Zubair, M. (2016). A new Weibull–Pareto distribution: properties and applications. *Communications in Statistics-Simulation and Computation*, *45*(10), 3548-3567.

24. Zea, L. M., Silva, R. B., Bourguinon, M., Santos, A. M., and Codeiro, G. M. (2012). The Beta Exponentiated Pareto Distribution with Application to Bladder Cancer Susceptibility, International Journal of Statistics and Probability 1: 8-19.

Appendix A

Theorem 1: Let $(\Omega, \mathrm{F}, \mathrm{P})$ be a probability space and let $[\mathrm{d}_1, \mathrm{d}_2]$ be an interval with $d_1 < d_2$ $d_1 = -\infty, d_2 = \infty$). Also suppose that a continuous random variable $X : \Omega \to [\mathrm{d}_1, \mathrm{d}_2]$ has distribution function F. Let $h_1(x)$ and $h_2(x)$ be two real functions continuous on $[\mathrm{d}_1, \mathrm{d}_2]$ such that $\frac{E[\mathrm{h}_1(X) | \mathrm{X} \geq \mathrm{x}]}{E[\mathrm{h}_2(X) | \mathrm{X} \geq \mathrm{x}]} = p(x)$ where $p(x)$ is real function and should be in simple form. Assume that, $h_1(x), h_2(x) \varepsilon C([\mathrm{d}_1, \mathrm{d}_2])$ $p(x) \varepsilon C^2([\mathrm{d}_1, \mathrm{d}_2])$ and F is strictly monotone function and twofold continuously differentiable on interval $[\mathrm{d}_1, \mathrm{d}_2]$. Assume that the equality $h_2(x) p(x) = h_1(x)$ has no real result inside of $[\mathrm{d}_1, \mathrm{d}_2]$. Then cdf "F "is obtained from $h_1(x), h_2(x)$ and $p(x)$ functions as $F(x) = \int_0^x K \left| \frac{p'(t)}{p(t) h_2(t) - h_1(t)} \right| \exp(-s(t)) dt$, where $s(t)$ is obtained from equation $s'(t) = \frac{p'(t) h_2(t)}{p(t) h_2(t) - h_1(t)}$ and K is a constant, selected to make