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A weak Galerkin finite element method for time fractional reaction-diffusion-convection problems with variable coefficients

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ABSTRACT

In this paper, a weak Galerkin finite element method for solving the time fractional reaction-convection diffusion problem is proposed. We use the well known L_1 discretization in time and a weak Galerkin finite element method on uniform mesh in space. Both continuous and discrete time weak Galerkin finite element method are considered and analyzed. The stability of the discrete time scheme is proved. The error estimates for both schemes are given. Finally, we give some numerical experiments to show the efficiency of the proposed method.

1. Introduction

In this paper, we will consider the weak Galerkin finite element method for the following time fractional diffusion equations with variable coefficients

$$\begin{cases} {}_0^C D_t{}^{\nu} u(x,t) - \nabla \cdot (K(x,t)\nabla u(x,t)) + \nabla \cdot (\mathbf{b}u(x,t)) + cu(x,t) = f(x,t) \quad \text{in} \quad Q_T = \Omega \times J, \\ u(x,0) = g(x), \quad x \in \overline{\Omega} \\ u(x,t) = 0 \quad \text{on} \quad \partial\Omega \times [0,T], \end{cases} \tag{1}$$

where Ω is a polygonal or a polyhedral domain in $\mathbb{R}^d (d = 2, 3)$ with boundary $\partial\Omega$ and $J = (0, T]$. The fractional derivative ${}_0^C D_t{}^{\nu} u(x,t)$ is the Caputo fractional derivative of order $\nu \in (0, 1]$ defined below. The given functions $f(x,t), g(x), \mathbf{b}(x)$ and $c(x)$ are smooth functions. For the stability of the problem, we assume that $\mathbf{b} \in [W^{1,\infty}(\Omega)]^d$ and there exists a positive constant a such that $c + \frac{1}{2}\nabla \cdot \mathbf{b} \geq a > 0$. Assume also that the matrix K is symmetric and positive definite in the sense that there is a constant $k > 0$ such that

$$kw^T w \leq w^T K w, \quad \forall w \in \mathbb{R}^d. \tag{2}$$

The problem (1) is well-posed [12], [8]. For easy presentation, we will only consider two-dimensional problems (i.e., $d = 2$). We shall denote by ${}_0^C D_t{}^{\nu} u(x,t)$ the Caputo fractional derivative of order $\nu \in (0, 1]$ with respect to the time t defined by [10], [36]

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$$ {}_0^C D_t^{\nu} u(x,t) = \frac{1}{\Gamma(1-\nu)} \int_0^t (t-s)^{1-\nu} \frac{\partial u}{\partial s} u(x,s)\, ds, \quad t>0. $$

The other commonly used fractional derivative is the Riemann-Lioville derivative defined as [10], [36]

$$ {}_0^R D_t^{\nu} u(x,t) = \frac{1}{\Gamma(n-\nu)} \frac{\partial^n}{\partial t^n} \int_0^t (t-s)^{n-1-\nu} u(x,s)\, ds, \quad t>0, \quad \nu \in (n-1,n]. \tag{3} $$

Over the last few decades, fractional calculus has been attracted an increasing attention and many important phenomena in various scientific areas such as physical and biological systems are described by fractional differential models [10], [36]. The fractional diffusion process is an important example in subdiffusion model where the diffusion is anomalously slow which can be modelled by fractional partial differential equation (1). Anomalous diffusion problems can be modelled by Brownian motion and Langevin equations with fractional time and variable diffusion coefficient which leads to a fractional power law scaling in time [7]. These problems can also be used in applications such as porous flows, biological processes and financial systems [19]. In a series of papers [24], [25], [26], [27], [29], [28], Ray studied and analyzed complex problems and systems governed by anomalous diffusion equation with Riesz fractional derivatives. In [24], the author developed numerical solutions of fractional Fokker–Planck equations with Riesz space fractional derivatives using shifted Grünwald approximation and fractional centred difference approaches in a finite domain. The author investigated the implicit finite difference scheme to approximate the Riesz fractional derivative and a novel modified optimal homotopy asymptotic method with Fourier transform to compute the numerical solution of Riesz fractional nonlinear Schrödinger equation in [25]. A time-splitting spectral approximation method for Chen-Lee-Liu equation with Riesz fractional derivative has been studied in [27] and it is proved that the proposed method is efficient, unconditionally stable and second-order of convergence rate in both time and space. Recently, the author proposed operational matrices based on two-dimensional Legendre wavelets for solving the variable-order fractional integro-differential equations and established the convergence analysis an error estimate for the method in [29]. The similarity method using fractional centred difference method and weighted shifted Grüwald-Letnikov difference method for reducing equations has been used for solving fractional Keller-Segel model with a nonlocal fractional Laplace operator which represents the fractional diffusion process [28].

When the diffusion coefficient $K(x,t)$ is constant, there exist many papers devoted to numerical approximations of the problem (1). A finite difference method for a diffusion-wave system by transforming the original equation into a low order system of equations with introducing new variables has been studied by Sun and Wu [30]. The finite element method with non smooth data has been considered by Jin et al. in [9]. Numerical methods in two dimension with variable-order fractional time derivative have been proposed in [2]. An implicit solution method has been analyzed in [11]. On the other hand, there are few papers in the literature for the numerical solution of the model problem (1) with variable diffusion coefficient because some difficulties arise due to the variable diffusivity. In *one-dimensional* case, a finite difference method with convergence rate of $\mathcal{O}(h^k+\tau^2)(k\in\{2,4\})$ for the time fractional diffusion equations has been studied in [1], a compact exponential numerical method with convergence rate of $\mathcal{O}(h^4+\tau^{2-\nu})$ for the time fractional convection-diffusion equations with variable coefficients has been proposed in [5], where τ is the temporal step size and h is space step size. For the time fractional diffusion problems with variable coefficients, Mustapha et al. [23] investigated a piecewise linear, time stepping discontinuous Galerkin with convergence of order $\mathcal{O}(h^2+\tau^{2-\frac{\nu}{2}})$ which order $\frac{\nu}{2}$ from being optimal in time due to the initial layer of the solution. In relation, Liu et al. [16], [17], [18] studied on integrability, analysis and generalized of the time fractional nonlinear diffusion problems. Yang [35] proposed a new integral transform operator to solve analytically the heat-diffusion problem when $\nu=1$. Yang et al. [37] discussed the analytical solutions of anomalous diffusion equations with the Caputo-Fabrizio fractional derivative in time with the help of the Laplace transform.

The main purpose of this paper is to study, both theoretically and computationally, the weak Galerkin finite element method for the time fractional reaction-diffusion-convection problem with variable coefficients. The weak Galerkin finite element method is a newly developed method based on the so called weak function spaces and discrete weak gradient operators. The main feature of this method lies in using completely discontinuous spaces for approximations in the framework of the finite element method. Because of this feature, the weak Galerkin finite element method is more suitable than the conventional finite element method for approximation of discontinuous solutions of problems on more complicated domains. Compared with the traditional discontinuous Galerkin methods which have sufficiently large penalty parameter for the stability, the weak Galerkin finite element method has parameter-free formulation. This method initially was proposed in [33], [34] and later has been used for solving parabolic equation [13], Biharmonic equation [31], [32], Helmholtz equation [20], [21], Darcy-Stokes equation [4] [15] and time-harmonic Maxwell's equations [22].

The rest of the paper is organized as follows. Section 2 introduces some notation, the discrete weak gradient, the discrete weak divergence and weak finite element spaces and the continuous time weak Galerkin finite element approximation for the problem (1). Section 3 contains a finite difference approximation of the fractional time derivative and the discrete time weak Galerkin finite element scheme and the stability of this scheme. In section 4, we establish the optimal order error estimates for the continuous and discrete time weak Galerkin method in L^2 norm. A new Gronwall type inequality which is the key ingredient in proving error estimate for the discrete time weak Galerkin scheme is also given in this section. We conclude the paper with some numerical examples to illustrate the theoretical results.

2. The weak Galerkin finite scheme

We use the standard Sobolev spaces $H^p(S)$, $p \geq 0$ for any polyhedron $S \subset \Omega$. We denote the associated inner product $(\cdot,\cdot)_{p,S}$, norms $\|\cdot\|_{p,S}$ and semi-norms $|\cdot|_{p,S}$ given by, respectively

$$\|u\|_{p,S} = \Big(\sum_{m=0}^{p} |u|_{m,S}^2\Big)^{1/2} \text{ and } |u|_{p,S} = \Big(\sum_{|\alpha|=p} \int_S |\partial^\alpha u|^2\, dx\Big)^{1/2},$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$, $|\alpha| = \sum_{j=1}^n \alpha_j$, $\partial^\alpha = \partial_{x_1}^{\alpha_1} \ldots \partial_{x_n}^{\alpha_n}$. When $p = 0$, the Hilbert space $H^p(S)$ will be the space of square integrable functions $L^2(S)$ and we suppress the subscript p in the inner product, semi-norm and norm notation in this case. Furthermore, we use the abbreviation $\|\cdot\|$ when $S = \Omega$ and $p = 0$.

Let $\mathcal{T}_h$ be a partition of the domain consisting of closed and simply connected polygon elements T. Let $\mathcal{E}_h$ be the set of all edges in $\mathcal{T}_h$, and $\mathcal{E}_h^0$ be the set of all interior edges and h_T be the diameter of element T with $h = \max_{T\in\mathcal{T}_h} h_T$.

We introduce the space of discrete weak functions and the discrete weak gradient operator introduced in [33]. Let $\mathbb{P}_k(T)$ be the set of polynomials on $T \in \mathcal{T}_h$ of degree at most k and $P_k(\partial T)$ be the space of polynomials on ∂T of degree at most k. A discrete weak function space $W(T,k)$ consists of weak functions $v = \{v_0, v_b\}$ on T such that $v_0 \in L^2(T)$ and $v_b \in L^\infty(\partial T)$. For given $k \geq 1$, let $S_h(k)$ be the weak Galerkin finite element space defined by

$$S_h(k) = \{v = \{v_0, v_b\} \in W(T,k) : v_0 \in \mathbb{P}_k(T),\, v_b \in \mathbb{P}_k(e) \text{ for all edge } e \subset \partial T, T \in \mathcal{T}_h\} \tag{4}$$

and a subspace $S_h^0(k)$ of $S_h(k)$ such that

$$S_h^0(k) = \{v : v \in S_h(k),\, v_b = 0 \text{ on } \partial\Omega\}. \tag{5}$$

We remark that the first component of v, namely v_0, is the value of v in the interior of T, and the second component v_b is the single value on the edges of the boundary of T which is not necessarily the trace of v_0 on the boundary.

For any $v = \{v_0, v_b\} \in S_h(k)$, we define the *weak gradient* $\nabla_w v \in [\mathbb{P}_{k-1}(T)]^2$ of v on T as the unique polynomial given by

$$(\nabla_w v, q)_T = -(v_0, \nabla\cdot q)_T + \langle v_b, q\cdot\mathbf{n}\rangle_{\partial T} \qquad \forall q \in [\mathbb{P}_{k-1}(T)]^2, \tag{6}$$

where $\mathbf{n}$ is the unit outward normal direction to ∂T. For any $v = \{v_0, v_b\} \in S_h(k)$, we define the *weak divergence* $\nabla_w \cdot (\mathbf{b}v) \in \mathbb{P}_k(T)$ of v related to $\mathbf{b}$ on T as the unique polynomial given by

$$(\nabla_w \cdot (\mathbf{b}v), z)_T = -(\mathbf{b}v_0, \nabla z)_T + \langle \mathbf{b}\cdot\mathbf{n} v_b, z\rangle_{\partial T} \quad \forall z \in \mathbb{P}_k(T). \tag{7}$$

In order to analyze and investigate the proposed method, we introduce element-wise defined four L^2 projections. We first define two local projections as: $Q_0 : L^2(T) \to \mathbb{P}_k(T)$ and $Q_b : L^2(\partial T) \to \mathbb{P}_k(\partial T)$ for each element $T \in \mathcal{T}_h$. The third one is the L^2 projection on the local weak gradient space defined as $\mathbb{Q}_h : [L^2(T)]^2 \to [\mathbb{P}_{k-1}(T)]^2$ for each element T. The fourth projection operator for the solution u is defined by $Q_h v = \{Q_0 v, Q_b v\} \in S_h(k)$.

For the sake of simplicity, we adopt the following notations,

$$\big(u, v\big) = \sum_{T\in\mathcal{T}_h} \big(u, v\big)_T = \sum_{T\in\mathcal{T}_h} \int_T uv\, dx,$$

$$\langle u, v\rangle = \sum_{T\in\mathcal{T}_h} \langle u, v\rangle_T = \sum_{T\in\mathcal{T}_h} \int_{\partial T} uv\, ds.$$

The standard weak form of (1) can be formulated, after multiplying by $v \in H_0^1(\Omega)$ and integration by parts, as follows

$$\begin{cases} \Big({}_0^C D_t{}^\nu u, v\Big) + \Big(K\nabla u, \nabla v\Big) - (\mathbf{b}u, \nabla v) + (cu, v) \;= (f, v), \quad \forall v \in H_0^1(\Omega), t \in J, \\ u(x, 0) = g(x), \quad x \in \overline{\Omega}. \end{cases} \tag{8}$$

Here, the solution u to the problem (8) is called weak solution and the existence and uniqueness of the weak solution for the problem (8) can be found in [14].

The semi-discrete continuous time weak Galerkin finite element method for the problem (1) can be formulated by replacing the standard gradient and divergence operator in the weak form (8) with the weak gradient and divergence operators as follows:

Algorithm 1 The weak Galerkin scheme for the time fractional convection-diffusion-reaction problem.

The WG-FEM for the problem (1) is to find $u_h(t) = \{u_0(.,t), u_b(.,t)\} \in S_h^0(k)$ satisfying $u_h(0) = Q_h g$ and the following equation:

$$\left({}_0^C D_t{}^{\nu} u_h(t), v_0\right) + a(u_h(t), v) = (f, v_0), \quad \forall v = \{v_0, v_b\} \in S_h^0(k), \tag{9}$$

where

$$a(u, v) = a_1(u, v) + s_c(u, v) + s_d(u, v), \tag{10}$$

with

$$\begin{aligned} a_1(u, v) &= (K\nabla_w u, \nabla_w v) + (\nabla_w \cdot (\mathbf{b}v), v_0) + (cu_0, v_0), \\ s_c(u, v) &= \sum_{T\in\mathcal{T}_h} (\mathbf{b}\cdot\mathbf{n}(u_0 - u_b), v_0 - v_b)_{\partial_+ T}, \\ s_d(u, v) &= \sum_{T\in\mathcal{T}_h} h_T^{-1}(u_0 - u_b, v_0 - v_b)_{\partial T}, \end{aligned}$$

and

$$\partial_+ T = \{y \in \partial T : \mathbf{b(y)}\cdot\mathbf{n(y)} \geq 0\}.$$

We have two stabilizer terms in our formulation of the bilinear form $a(\cdot,\cdot)$. The first stabilizer $s_c(\cdot,\cdot)$ for convection term has simpler and upwinding-type structure while the second stabilizer $s_d(\cdot,\cdot)$ is controlling the jump between u_0 and u_b on the boundary of the element T. Note that unlike the most existing discontinuous Galerkin methods that assume the convection term $\mathbf{b}$ is either constant or divergence-free and the weak Galerkin method proposed in [3], our method does not insist on extra requirements.

We accordingly define the energy norm for any $u \in S_h^0(k)$ as follows

$$|||u|||^2 = \sum_{T\in\mathcal{T}_h} \|\nabla_w u\|_T^2 + \sum_{T\in\mathcal{T}_h} \left\| |\mathbf{b}\cdot\mathbf{n}|^{1/2}(u_0 - u_b)\right\|_{\partial T}^2 + \|u_0\|^2 + s_d(u, u). \tag{11}$$

The bilinear form $a(\cdot,\cdot)$ is continuous and coercive with respect to the norm given by (11).

Lemma 2.1. *For $u, v \in S_h^0(k)$, there are constants $C_1, C_2 > 0$ such that*

$$a(u, v) \leq C_1 |||u|||\,|||v|||, \tag{12}$$

$$a(u, u) \geq C_2 |||u|||^2. \tag{13}$$

Proof. The continuity (12) of the bilinear form follows easily from the definition of the bilinear form and the norm. We show the coercivity (13) of the bilinear form. It follows from the definition of the weak divergence (7) and integration by parts that for any $v = \{v_0, v_b\} \in S_h^0(k)$

$$\begin{aligned} (\nabla_w \cdot (\mathbf{b}v), v_0) &= (-\mathbf{b}v_0, \nabla v_0) + \langle \mathbf{b}\cdot\mathbf{n}v_b, v_0\rangle \\ &= (\nabla\cdot\mathbf{b}v_0, v_0) + (\mathbf{b}v_0, \nabla v_0) - \langle \mathbf{b}\cdot\mathbf{n}(v_0 - v_b), v_0\rangle \\ &= (\nabla\cdot\mathbf{b}v_0, v_0) - (\nabla_w\cdot(\mathbf{b}v), v_0) + \langle \mathbf{b}\cdot\mathbf{n}v_b, v_0\rangle - \langle \mathbf{b}\cdot\mathbf{n}(v_0 - v_b), v_0\rangle \\ &= (\nabla\cdot\mathbf{b}v_0, v_0) - (\nabla_w\cdot(\mathbf{b}v), v_0) - \langle \mathbf{b}\cdot\mathbf{n}(v_0 - v_b), v_0 - v_b\rangle, \end{aligned}$$

which implies that

$$(\nabla_w \cdot (\mathbf{b}v), v_0) = \frac{1}{2}(\nabla\cdot\mathbf{b}v_0, v_0) - \frac{1}{2}\langle \mathbf{b}\cdot\mathbf{n}(v_0 - v_b), v_0 - v_b\rangle, \tag{14}$$

where we used the facts that $\langle \mathbf{b}\cdot\mathbf{n}v_b, v_b\rangle = 0$.

Using (2) and (14), we have

$$
\begin{aligned}
a(v,v) &= (K\nabla_w v, \nabla_w v) + \left(\left(c+\frac{1}{2}\nabla\cdot\mathbf{b}\right)v_0, v_0\right) - \frac{1}{2}\langle \mathbf{b}\cdot\mathbf{n}(v_0-v_b), v_0-v_b\rangle + s_c(v,v) + s_d(v,v)\\
&\geq k(\nabla_w v, \nabla_w v) + a(v_0,v_0) + \frac{1}{2}\sum_{T\in\mathcal{T}_h}\left\| |\mathbf{b}\cdot\mathbf{n}|^{1/2}(u_0-u_b)\right\|_{\partial T}^2 + s_d(v,v)\\
&\geq C_2 |||v|||^2,
\end{aligned}
$$

where $C_2 = \min\{k, a, \frac{1}{2}\}$. We complete the proof. ■

3. A finite difference method for time discretization

We first discretize the Caputo time fractional derivative by using a finite difference method. Let $t_m := m\tau,\ m = 0,1,\dots,M$ where $\tau := \frac{T}{M}$ is the time step. Let $U^m \in S_h(k)$ be the numerical solution of $u(t_m)$. Based on a piecewise linear interpolation, a numerical scheme of the Caputo fractional derivative is given by

$$
\begin{aligned}
{}_0^C D_{t_m}^{\nu} u(x,t_m) &= \frac{1}{\Gamma(1-\nu)}\int_0^{t_m}(t_m-s)^{-\nu}\frac{\partial u}{\partial s}u(x,s)\,ds\\
&= \frac{1}{\Gamma(1-\nu)}\sum_{k=1}^{m}\frac{u(x,t_k)-u(x,t_{k-1}}{\tau}\int_{t_{k-1}}^{t_k}(t_m-s)^{-\nu}\,ds + R^m\\
&= \frac{\tau^{-\nu}}{\Gamma(2-\nu)}\sum_{k=1}^{m} b_{m-k}(u(x,t_k)-u(x,t_{k-1})) + R^m \qquad (15)\\
&= \frac{\tau^{-\nu}}{\Gamma(2-\nu)}\left(b_0 u(x,t_m) - \sum_{k=1}^{m-1}(b_{m-k-1}-b_{m-k})u(x,t_k) - b_{m-1}u(x,t_0)\right) + R^m \qquad (16)\\
&:= L_t^{\nu} u + R^m \qquad (17)
\end{aligned}
$$

where $b_j = (j+1)^{1-\nu} - j^{1-\nu},\ j \geq 0$. The truncation error R^k has the following expressing.

Theorem 3.1. *[30] If $u \in C^2[0,T]$, then the truncation error R^k satisfies*

$$
|R^m| \leq C\tau^{2-\nu}\max_{t\in[0,t_m]}|u''(t)|, \qquad (18)
$$

where $C = \frac{1}{1-\nu}(\frac{1-\nu}{12} + \frac{2^{2-\nu}}{2-\nu} - (1+2^{-\nu}))$.

Next, we consider the fully discrete time weak Galerkin finite element scheme for (8). Find $U^m \in S_h(k)$ such that

$$
(L_t^{\nu} U_0^m, v_0) + a(U^m, v) = (f^m, v_0), \quad \forall v \in S_h^0(k). \qquad (19)
$$

Using the approximation L_t^{ν} in (17), we can rewrite the scheme (19) as

$$
(U_0^m, v_0) + c_0 a(U^m, v) = \sum_{k=1}^{m-1}(b_{m-k-1}-b_{m-k})(U_0^k, v_0) + b_{m-1}(U_0^0, v_0) + (f^m, v_0) \qquad (20)
$$

where $c_0 = \tau^{\nu}\Gamma(2-\nu)$ and $f^m = f(x,t_m)$. Using the following properties of the coefficients b_j

$$
\begin{aligned}
& b_j > 0, \quad j = 0,1,\dots,m,\\
& b_1 > b_2 > \dots > b_m, \quad b_0 = 1, \quad b_m \to \infty,\\
& \sum_{j=1}^{m+1}(b_{j-1}-b_j) + b_{m+1} = (1-b_1) + \sum_{j=2}^{m}(b_{j-1}-b_j) + b_m = 1,
\end{aligned} \qquad (21)
$$

we will prove the stability of the time discrete weak Galerkin method (20) in the next theorem. We will assume $f \equiv 0$ in proving the stability estimate in the next theorem for simplicity.

Theorem 3.2. *The fully time discrete weak Galerkin finite element scheme (20) is unconditionally stable and we have*

$$||U_0^m|| \le (\frac{1}{1+c_0C_2})^{1/2}\|U_0^0\| \tag{22}$$

where $c_0 = \Gamma(2-\nu)\tau^\nu$ *and* C_2 *is given in (13).*

Proof. The result will be proved by induction. For $m = 1$, taking $v = U^1$ in (20), we have

$$(U_0^1, U_0^1) + c_0 a(U^1, U^1) = (U_0^0, U_0^1).$$

Using Lemma 2.1 and Cauchy-Schwarz inequality, we have

$$||U_0^1|| \le \frac{1}{1+c_0C_2}\|U_0^0\|.$$

Assume that (22) holds for $m = 1, \dots, M-1$. We will show that (22) holds also for $m = M$. Choosing $v = U^M$ in (20) leads to

$$\begin{aligned}(U_0^M, U_0^M) + c_0 a(U^M, U^M) &= \sum_{k=1}^{M-1}(b_{M-k-1} - b_{M-k})(U_0^k, U_0^M) + b_{m-1}(U_0^0, U_0^M)\\ &\le \sum_{k=1}^{M-1}(b_{M-k-1} - b_{M-k})\frac{\|U_0^k\|^2 + \|U_0^M\|^2}{2} + b_{m-1}\frac{\|U_0^0\|^2 + \|U_0^M\|^2}{2}\\ &= \sum_{k=1}^{M-1}(b_{M-k-1} - b_{M-k})\|U_0^k\|^2 + b_{m-1}\|U_0^0\|^2.\end{aligned}$$

Using the induction assumption, the properties of the coefficients (21) and Lemma 2.1, we obtain

$$\|U_0^M\|^2 \le \frac{1}{1+c_0C_2}\Big(\sum_{k=1}^{M-1}(b_{M-k-1} - b_{M-k}) + b_{m-1}\Big)\|U_0^0\|^2 = \frac{1}{1+c_0C_2}\|U_0^0\|^2.$$

Thus, the proof is completed by taking the square root of both sides. ∎

4. Error analysis

This section deals with the optimal order error estimates for the continuous weak Galerkin finite element scheme (9) and discrete time weak Galerkin finite element method (19) in L^2 norm. First, we define the elliptic or Ritz projection $R_h u$ of $u \in H_0^1(\Omega) \cap H^2(\Omega)$ onto the discrete weak space S_h^0 satisfying the equation

$$a(R_h u, v) = (-\nabla\cdot(K\nabla u), v) + (\nabla\cdot(\mathbf{b}u), v) + (u, v), \quad \forall v \in S_h^0(k). \tag{23}$$

In fact $R_h u$ is the weak Galerkin finite element solution of the corresponding elliptic problem:

$$\begin{aligned}-\nabla\cdot(K\nabla u) + \nabla\cdot(\mathbf{b}u) + cu = F \quad &\text{in} \quad \Omega,\\ u = 0 \quad &\text{on} \quad \partial\Omega.\end{aligned} \tag{24}$$

We then have the following error estimate for $R_h u$.

Lemma 4.1. *[13] Let* $u \in H^{1+k}(\Omega)$ *be the exact solution of (24) and* $R_h u$ *be the elliptic projection defined by (23). Let* $Q_h u = \{Q_0 u, Q_b u\}$*. Then there exists a constant C such that*

$$\|Q_0 u - R_h u\| \le Ch^{k+1}\|u\|_{k+1}, \tag{25}$$

and

$$\|\nabla_w(Q_h u - R_h u)\| \le Ch^k\|u\|_{k+1}. \tag{26}$$

We first prove an error estimate for the continuous time semidiscrete problem of (9). For the sake of simplification and to avoid the long algebra, we consider the zero initial condition, i.e., $g \equiv 0$ for the continuous time weak Galerkin finite element approximation.

It is well known that the Riemann-Lioville and the Caputo fractional derivatives agree if the initial condition is zero, i.e., the following relation holds true [6]

$$ {}_0^R D_t{}^{\nu} u(t) = {}_0^C D_t{}^{\nu} u(t), \quad \text{if} \quad u(0) = 0, \quad \nu \in (0, 1]. \tag{27} $$

The following lemma will be useful in the sequel.

Lemma 4.2. *[6] If $\nu \in (0, 1)$ and $u \in H^{\nu}((0, T), L^2(\Omega))$, then we have*

$$ \int_0^T \left({}_0^R D_t{}^{\nu} u(t), u(t) \right)_{L^2(\Omega)} dt = \int_0^T \left({}_0^R D_t{}^{\nu/2} u(t), {}_0^R D_t{}^{\nu/2} u(t) \right)_{L^2(\Omega)} dt. $$

We are now ready to state and prove the error estimates for the continuous time weak Galerkin finite element scheme (9) in L^2 norm.

Theorem 4.3. *Suppose that u and u_h are the solutions of the problem (1) and the continuous time weak Galerkin finite element scheme (9), respectively. Let $e = u_h - Q_h u$ be the error between the continuous time weak Galerkin approximation and the L^2 projection of the exact solution u. Then there is a positive constant C such that*

$$ \int_0^T \| {}_0^C D_t{}^{\nu/2} e \|_{L^2(\Omega)}^2 \, dt \le C h^{2(k+1)} \int_0^T \| {}_0^C D_t{}^{\nu/2} u \|_{k+1}^2 \, dt. \tag{28} $$

Proof. We write

$$ e = \theta(t) + \rho(t), \quad \text{where} \quad \theta(t) = u_h - R_h u, \quad \rho(t) = R_h u - Q_h u. $$

The error term ρ can be easily bounded by Lemma 4.1 as follows

$$ \int_0^T \| {}_0^C D_t{}^{\nu/2} \rho(t) \|_{L^2(\Omega)}^2 \, dt \le C h^{2(k+1)} \int_0^T \| {}_0^C D_t{}^{\nu/2} u \|_{k+1}^2 \, dt. \tag{29} $$

We will estimate θ. Using the definition of bilinear form, we have for any $v \in S_h^0(k)$

$$ \begin{aligned} ({}_0^C D_t{}^{\nu} \theta, v) + a(\theta, v) &= ({}_0^C D_t{}^{\nu} u_h, v) + a(u_h, v) - ({}_0^C D_t{}^{\nu} R_h u, v) - a(R_h u, v) \\ &= (f, v) - ({}_0^C D_t{}^{\nu} R_h u, v) - a(R_h u, v) \\ &= ({}_0^C D_t{}^{\nu} u, v) - ({}_0^C D_t{}^{\nu} R_h u, v) = ({}_0^C D_t{}^{\nu} \rho, v), \end{aligned} $$

which implies that

$$ ({}_0^C D_t{}^{\nu} \theta, v) + a(\theta, v) = ({}_0^C D_t{}^{\nu} \rho, v), \quad \forall v \in S_h^0(k). \tag{30} $$

Here, we used the fact that the projection R_h commutes with the fractional derivative. Choosing $v = \theta \in S_h^0(k)$ in the above equation (30) and integrating on $[0, T]$, we have

$$ \int_0^T ({}_0^C D_t{}^{\nu} \theta, \theta) \, dt + \int_0^T a(\theta, \theta) \, dt = \int_0^T ({}_0^C D_t{}^{\nu} \rho, \theta) \, dt. $$

Using the relation (27), Lemma 4.2 and the coercivity of the bilinear form $a(\cdot, \cdot)$ in (2.1) along with the Arithmetic-Geometric mean inequality, we arrive at

$$ \begin{aligned} \int_0^T \| {}_0^R D_t{}^{\nu/2} \theta \|_{L^2(\Omega)}^2 \, dt + C \int_0^T |||\theta|||^2 \, dt &= \int_0^T ({}_0^R D_t{}^{\nu/2} \rho, {}_0^R D_t{}^{\nu/2} \theta) \, dt \\ &\le \int_0^T C_\delta \| {}_0^R D_t{}^{\nu/2} \rho \|_{L^2(\Omega)}^2 \, dt + \delta \int_0^T \| {}_0^R D_t{}^{\nu/2} \theta \|_{L^2(\Omega)}^2 \, dt \end{aligned} $$

for sufficiently small $\delta > 0$. Then we obtain

$$\int_0^T \|{}_0^R D_t{}^{\nu/2}\theta\|^2_{L^2(\Omega)}\,dt \le C\int_0^T \|{}_0^R D_t{}^{\nu/2}\rho\|^2_{L^2(\Omega)}\,dt \tag{31}$$

Using the bound (29), we have

$$\int_0^T \|{}_0^R D_t{}^{\nu/2}\theta\|^2_{L^2(\Omega)}\,dt \le Ch^{2(k+1)}\int_0^T \|{}_0^C D_t{}^{\nu/2}u\|^2_{k+1}\,dt. \tag{32}$$

Finally, combining the estimates (29) and (32) gives the desired result. Thus we complete the proof. ■

We next prove the error estimates for the discrete time weak Galerkin finite element scheme (20) in L^2-norm. The following lemma which establishes a new Gronwall-type inequality will be used in proving the error estimate.

Lemma 4.4. *[9] Suppose that the nonnegative sequences $\{u_n, g_n : n \in \mathbb{N}\}$ satisfy*

$$L_t^\nu u_n \le \lambda_1 u_n + \lambda_2 u_{n-1} + g_n, \quad n \ge 1,$$

where λ_1, λ_2 is nonnegative constants. Then there is a positive constant τ^* such that, when $\tau \le \tau^*$, we have

$$u_n \le 2\Big(u_0 + \frac{t_n^\nu}{\Gamma(1+\nu)}\max_{0\le k\le n} g_k\Big)E_\nu(2\lambda t_n^\nu), \quad n \in [1, N], \tag{33}$$

where $E_\nu(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(1+k\nu)}$ is the Mittag-Leffler function and $\lambda = \lambda_1 + \frac{\lambda_2}{(2-2^{1-\nu})}$.

Theorem 4.5. *Assume that $u(t_m)$ is the solution of the problem (1) and U^m is the solution of the discrete time weak Galerkin scheme (20). Then there is a positive constant t_0 such that for all $t_m \le t_0$*

$$\|U^m - Q_h u(t_m)\| \le C\Big(\|U^0 - Q_h u(0)\| + Ch^{k+1}\Big(\|u_t(0) - Q_0 u_t(0)\| + \|u_t(m) - Q_0 u_t(m)\| + \|g\|_{k+1} + \int_0^{t_m} \|u_t\|_{k+1} + \|{}_0^C D_{t_m}^\nu u\|_{k+1}\Big) + \frac{T^\nu \tau^{2-\nu}}{\Gamma(1+\nu)}\max_{0\le t\le t_m}|\frac{\partial^2 u}{\partial t^2}|\Big).$$

Proof. The exact solution $u^m = u(t_m)$ satisfies the following equation

$$(L_t^\nu u^m, v_0) + a(u^m, v) = (f^m, v_0) + (R^m, v_0), \quad \forall v \in S_h^0(k) \tag{34}$$

where R^m is the truncation error given by (18).

As before, we let $e^m = Q_h u(t_m) - U^m = R_h u(t_m) - U^m + Q_h u(t_m) - R_h u(t_m) = \theta^m + \rho^m$. Using Lemma 4.1, we can estimate $\rho^m = \rho(t_m)$ as

$$\|\rho(t_m)\| = Ch^{k+1}(\|u_t(m) - Q_0 u_t(m)\| + \|g\|_{k+1} + \int_0^{t_m} \|u_t\|_{k+1}). \tag{35}$$

Subtracting the equation (34) from the numerical scheme (19) gives

$$(L_t^\nu \theta_0^m, v_0) + a(\theta^m, v) = (L_t^\nu(R_h u(t_m) - u^m), v_0) + (R^m, v_0), \quad \forall v \in S_h^0(k). \tag{36}$$

Taking $v = \theta^m$ in the above equation (36), we have

$$\begin{aligned}(L_t^\nu \theta^m, \theta^m) + a(\theta^m, \theta^m) &= (L_t^\nu(R_h u(t_m) - u^m), \theta^m) + (R^m, \theta^m)\\ &\le \frac{1}{2}\|L_t^\nu(R_h u(t_m) - u^m)\|^2 + \|\theta^m\|^2 + \frac{1}{2}\|R^m\|^2\\ &\le \|\theta^m\|^2 + C\Big(\tau^{2-\nu}\max_{0\le t\le t_m}|\frac{\partial^2 u}{\partial t^2}| + h^{k+1}\|{}_0^C D_{t_m}^\nu u\|_{k+1}\Big)^2,\end{aligned} \tag{37}$$

where we have used the remainder error (18) and the fact that

$$\begin{aligned}\|L_t^{\nu} R_h u(t_m) - {}_0^C D_{t_m}{}^{\nu} u\| &\le \|L_t^{\nu} R_h u(t_m) - {}_0^C D_{t_m}^{\nu} R_h u\| + \|{}_0^C D_{t_m}^{\nu} R_h u - {}_0^C D_{t_m}^{\nu} u\| \\ &\le C\Big(\tau^{2-\nu} \max_{0\le t\le t_m} |\frac{\partial^2 u}{\partial t^2}| + h^{k+1} \|{}_0^C D_{t_m}^{\nu} u\|_{k+1}\Big).\end{aligned}$$

Moreover, using the properties of coefficients b_j given in (21), we obtain

$$\begin{aligned}(L_t^{\nu}\theta^m, \theta^m) &= \frac{\tau^{-\nu}}{\Gamma(2-\nu)}\Big(b_0\theta^m - \sum_{k=1}^{m-1}(b_{m-k-1} - b_{m-k})e^k - b_{m-1}e^0, \theta^m\Big) \\ &\ge \frac{\tau^{-\nu}}{\Gamma(2-\nu)}\Big(b_0\|\theta^m\|^2 - \sum_{k=1}^{m-1}(b_{m-k-1} - b_{m-k})\frac{\|e^k\|^2 + \|\theta^m\|^2}{2} - b_{m-1}\frac{\|e^0\|^2 + \|\theta^m\|^2}{2}\Big) \\ &= \frac{\tau^{-\nu}}{2\Gamma(2-\nu)}\Big(b_0\|\theta^m\|^2 - \sum_{k=1}^{m-1}(b_{m-k-1} - b_{m-k})\|e^k\|^2 - b_{m-1}\|e^0\|^2\Big) \\ &= \frac{1}{2}L_t^{\nu}\|\theta^m\|^2.\end{aligned} \tag{38}$$

From (37) and (38), we have

$$L_t^{\nu}\|\theta^m\|^2 \le \|\theta^m\|^2 + C\Big(\tau^{2-\nu} \max_{0\le t\le t_m} |\frac{\partial^2 u}{\partial t^2}| + h^{k+1}\|{}_0^C D_{t_m}^{\nu} u\|_{k+1}\Big)^2.$$

By Lemma 4.4, there is a positive constant τ^* such that for all $\tau \le \tau^*$ it holds that

$$\|\theta^m\|^2 \le C\Big(\|\theta^0\|^2 + \frac{T^{\nu}}{\Gamma(1+\nu)}\Big(\tau^{2-\nu} \max_{0\le t\le t_m} |\frac{\partial^2 u}{\partial t^2}| + h^{k+1}\|{}_0^C D_{t_m}^{\nu} u\|_{k+1}\Big)^2\Big).$$

With the aid of Lemma 4.1, we estimate $\|\theta^0\| = \|\theta(0)\|$ as follows

$$\begin{aligned}\|\theta(0)\| = \|U^0 - R_h u(0)\| &\le \|U^0 - Q_h u(0)\| + \|R_h u(0) - Q_h u(0)\| \\ &\le \|U^0 - Q_h u(0)\| + Ch^{k+1}(\|u_t(0) - Q_0 u_t(0)\| + \|g\|_{k+1}).\end{aligned}$$

Combining (35) and the last inequalities complete the proof. ■

5. Numerical experiments

In this section, we give some numerical results to present the error e between the numerical solution u_h and the projection $Q_h u$. We use a uniform triangulation mesh $\mathcal{T}_h$ and the discrete weak space $S_h(0)$. That is, piecewise constants on the triangles and their edges will be used. For simplicity, we take $\Omega = [0,1]$, $T = 1$ and L^2 norm errors are denoted by $E(h,\tau) = \|U^m - Q_h u(t_m)\|$ and orders of convergence in space (OCS) and in time (OCT) calculated by the formula $OCS = \frac{\log \frac{E(2h,\tau)}{E(h,\tau)}}{\log 2}$ and $OCT = \frac{\log \frac{E(h,2\tau)}{E(h,\tau)}}{\log 2}$, respectively.

Example 1. Consider the following time fractional diffusion equation in $Q = [0,1] \times [0,1]$,

$$\begin{aligned}{}_0^C D_t{}^{\nu} u - \nabla\cdot(K\nabla u) + \nabla\cdot(\mathbf{b}u) + cu = f, &\quad \text{in} \quad Q, \quad t > 0, \\ u = 0 &\quad \text{on} \quad \partial Q, \quad t > 0, \\ u(x,0) = g(x), &\quad \text{in} \quad Q,\end{aligned} \tag{39}$$

where $K = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\mathbf{b} = (1,1)^T$, $c = 0$ and f and g are chosen so that the exact solution is given by

$$u(x,y,t) = (1+t^3)\sin(2\pi x)\sin(2\pi y).$$

We first fix $\tau = 1/1000$, $T = 1$ and take the mesh size $h = \frac{1}{2^n}$, $n = 1,2,3,4,5,6$. L^2 errors $E(h,\tau)$ and orders of convergence OCS are shown in Table 1. We also plot the L^2 norm errors for various values of ν in Fig. 1 and Fig. 2

We next fix $h = 1/1000$ and change the time step size $\tau = \frac{1}{2^n}$, $n = 1,2,3,4,5,6$. The errors $E(h,\tau)$ and orders of convergence OCT for $\nu = 0.5, 0.75, 0.90$ are shown in Table 2.

Table 1
L^2 errors and OCS of the weak Galerkin finite element for Example 1 at $T = 1$ for a fixed $\tau = \frac{1}{1000}$.

h	$\nu = 0.5$		$\nu = 0.75$		$\nu = 0.90$	
	$E(h)$	OCS	$E(h)$	OCS	$E(h)$	OCS
1/2	4.65e-02	-	2.01e-02	-	7.45e-03	-
1/4	2.30e-02	1.0155	1.01e-02	1.0071	3.67e-03	1.0214
1/8	1.13e-02	1.0253	4.96e-03	1.0259	1.81e-03	1.0197
1/16	5.62e-03	1.0076	2.43e-03	1.0293	8.92e-04	1.0208
1/32	2.77e-03	1.0206	1.19e-03	1.0299	4.39e-04	1.0228
1/64	1.36e-03	1.0262	5.84e-04	1.0269	2.16e-04	1.0231

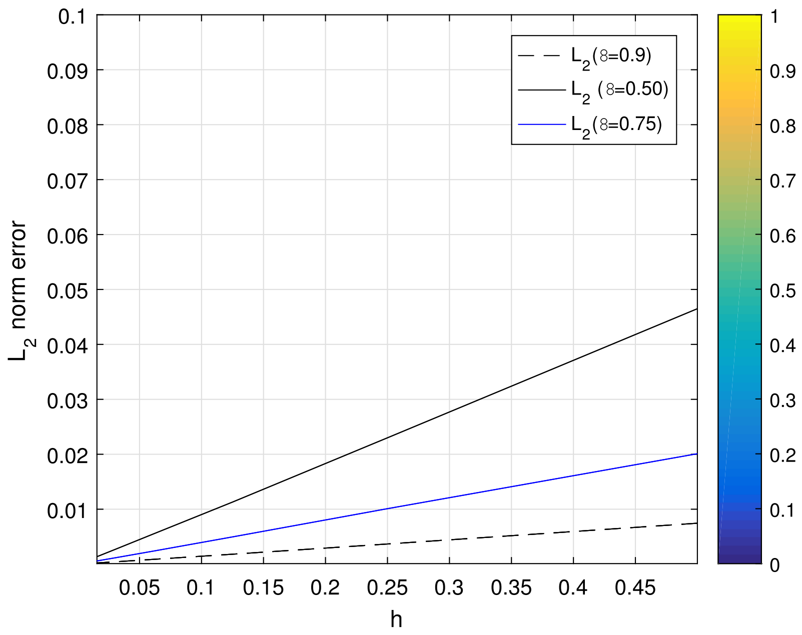

Fig. 1. Plots for L^2 error $E(h, \tau)$ for the different values of ν with the fixed time mesh for Example 1. (For interpretation of the colours in the figure(s), the reader is referred to the web version of this article.)

Table 2
L^2 errors and OCT of the weak Galerkin finite element for Example 1 at $T = 1$ for fixed $h = \frac{1}{1000}$.

τ	$\nu = 0.5$		$\nu = 0.75$		$\nu = 0.90$	
	$E(\tau)$	OCT	$E(\tau)$	OCT	$E(\tau)$	OCT
1/2	3.18e-02	-	8.08e-03	-	3.52e-03	-
1/4	1.12e-02	1.5055	3.25e-03	1.3139	1.65e-03	1.0931
1/8	3.95e-03	1.5035	1.31e-03	1.3108	7.71e-04	1.0976
1/16	1.38e-03	1.5171	5.29e-04	1.3082	3.60e-04	1.0987
1/32	4.86e-04	1.5056	2.15e-04	1.2989	1.68e-04	1.0995
1/64	1.71e-04	1.5069	8.72e-05	1.3019	7.82e-05	1.1032

Table 3
L^2 errors and OCS of the weak Galerkin finite element for Example 2 at $T = 1$ for fixed $\tau = \frac{2}{10000}$.

h	$\nu = 0.5$		$\nu = 0.75$		$\nu = 0.90$	
	$E(h)$	OCS	$E(h)$	OCS	$E(h)$	OCS
1/4	6.21e-01	-	3.12e-01	-	1.08e-01	-
1/8	2.98e-01	1.0592	1.55e-01	1.0092	5.38e-02	1.0053
1/16	1.46e-01	1.0293	7.71e-02	1.0074	2.67e-02	1.0107
1/32	7.28e-02	1.0039	3.84e-02	1.0056	1.31e-02	1.0272

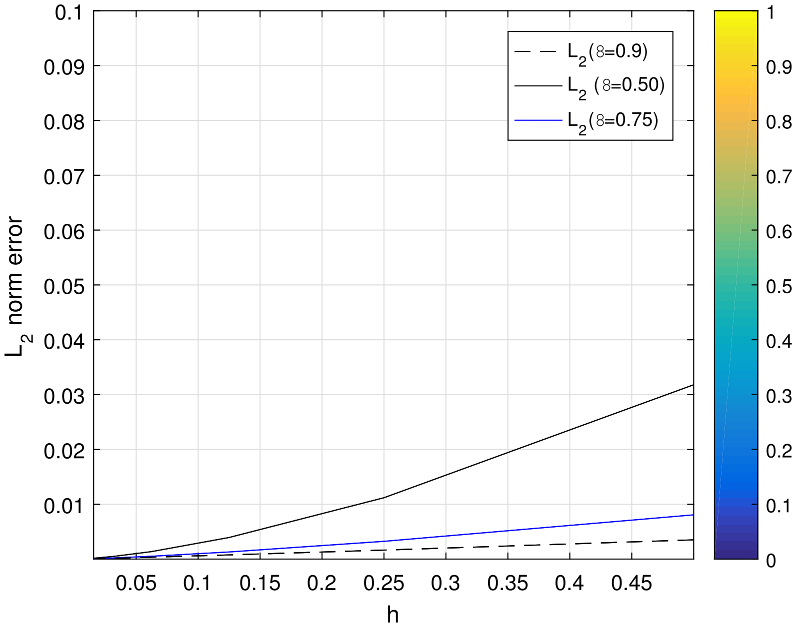

Fig. 2. Plots for L^2 error of $E(h,\tau)$ for the different values of ν with the fixed space mesh for Example 1.

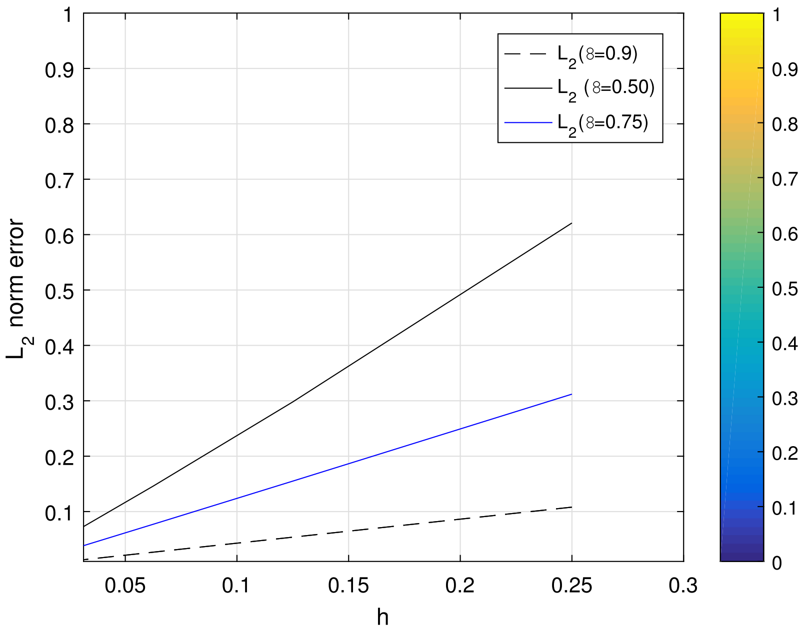

Fig. 3. Plots for L^2 error $E(h,\tau)$ for the different values of ν with the fixed time mesh for Example 2.

Example 2. Consider the following problem [13] in $Q=[0,1]\times[0,1]$,

$$\begin{aligned} {}_0^C D_t{}^{\nu} u - \nabla\cdot(K\nabla u) + \nabla\cdot(\mathbf{b}u) + cu = f, \quad &\text{in} \quad Q, \quad t>0, \\ u = 0 \quad \text{on} \quad \partial Q, \quad & t>0, \\ u(x,0) = 0, \quad \text{in} \quad Q, \end{aligned} \tag{40}$$

where $K=\begin{bmatrix} x^2+y^2+1 & xy \\ xy & x^2+y^2+1 \end{bmatrix}$, $\mathbf{b}=(1,1)^T$, $c=0$ and f is chosen so that the exact solution is given by

$$u(x,y,t)=\sin(2\pi t)\sin(2\pi x)\sin(2\pi y).$$

We take $\tau = 2/10000$, $T = 1$ and the mesh size $h = \frac{1}{2^n}$, $n = 2, 3, 4, 5$. L^2 errors and orders of convergence (OC) are shown in Table 3. We also plot the L^2 norm errors for various values of ν in Fig. 3.

6. Conclusion

In this manuscript, we consider the weak Galerkin finite element method for solving the time fractional reaction-diffusion-convection equations with variable coefficients. The semi-discrete weak Galerkin scheme in time and the fully discrete weak Galerkin finite element method using the standard $L1$- approximation in time is developed. Stability and error analysis is established. Various numerical examples are presented to show the efficacy of the proposed method. We show numerically that the results are consistent with the theoretical analysis.

Appendix A. Supplementary material

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.apnum.2021.05.021.

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