

Comparison of the Estimation Methods for the Parameters of Exponentiated Reduced Kies Distribution

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Abstract: In this paper, we consider the estimation for the parameters of exponentiated reduced Kies (ERK) distribution using maximum likelihood (ML), least squares (LS), weighted least squares (WLS), Cramér-von Mises (CM), Anderson Darling (AD) and right-tail Anderson Darling (RAD) methods. The performances of the estimators are compared via Monte-Carlo simulation study for different parameter settings and different sample sizes. Finally, a real data set is analyzed for the implementation of the proposed methods.

Exponentiated Reduced Kies Dağılımının Parametreleri için Tahmin Yöntemlerinin Karşılaştırılması

Anahtar Kelimeler

Exponentiated reduced Kies dağılımı,
Parametre tahmini,
Monte-Carlo simülasyonu,
Etkinlik

Özet: Bu makalede, exponentiated reduced Kies (ERK) dağılımının parametreleri en çok olabilirlik, en küçük kareler, ağırlıklandırılmış en küçük kareler, Cramér-von Mises, Anderson Darling ve sağ-kuyruklu Anderson Darling yöntemleri kullanılarak tahmin edilmiştir. Tahmin edicilerin performansları farklı parametre değerleri ve farklı örneklem boyutları için Monte-Carlo simülasyon çalışması ile karşılaştırılmıştır. Son olarak, önerilen yöntemlerin uygulanması için gerçek bir veri seti analiz edilmiştir.

1. Introduction

Weibull distribution is one of the most popular distributions used in engineering, life testing and natural science with its wide variety of the shapes [1]. However, Weibull distribution may not be appropriate for all data sets [2]. Especially, if the data represents unimodal or bathtub shaped hazard function, the usage of Weibull distribution may cause misspecification. To overcome this difficulty, there have been significant number of studies considering the extensions or modifications of Weibull models, such as exponentiated Weibull [3], generalized Weibull [4], modified Weibull [5], beta Weibull [6], beta exponentiated Weibull [7] distributions and so on.

In addition to these studies, [8] considered a functional form of Weibull distribution which is called as Kies distribution. The detailed information about this functional form and the distributional properties of Kies distribution are investigated by

[9]. They reported that Kies distribution have increasing, decreasing and bathtub-shaped hazard function. This property provides flexibility for modeling data. Furthermore, [10] proposed a new distribution, namely reduced Kies (RK) distribution which is a special case of Kies distribution. The exponential form of RK distribution is introduced by [11] and is called as exponentiated reduced Kies (ERK) distribution. They investigated some distributional properties of ERK distribution, obtained the maximum likelihood (ML) estimators and discussed asymptotic properties of these estimators. They modeled four different engineering data sets with ERK distribution and compared its modeling performance with different statistical distributions. They represented that the ERK distribution has better modeling performance than the commonly used statistical models.

In this paper, we consider the estimation of the parameters of ERK distribution by using ML, least squares (LS), weighted least squares (WLS), Cramér-

von Mises (CM), Anderson Darling (AD) and right-tailed Anderson Darling (RAD) methods. It should be noted that among from these methods, the CM, AD and RAD methods are called as minimum distance methods. The novelty of this paper comes from the fact that we compare the performances of classical and minimum distance estimation methods for estimating the parameters of ERK distribution.

The rest of this paper is organized as follows. In Section 2, we give brief description about ERK distribution and discuss the estimation methods used in this paper. Section 3 presents Monte-Carlo simulation study in order to identify the most efficient estimation methods and also we analyze a real data set to make an implementation of these methodologies. Final comments and conclusions are given in Section 4.

2. Material and Methods

In this section, we give some brief description about ERK distribution. Also, we consider the estimation for the parameters of ERK distribution by using six different estimation methods.

2.1. Exponentiated reduced Kies distribution

Let X be a random variable from ERK distribution with parameters α and β . The cumulative distribution function (cdf), probability density function (pdf), survival function and hazard function of X are defined as follows

$$F(x) = \left[1 - e^{-\left(\frac{x}{1-x}\right)^\alpha}\right]^\beta, \quad 0 < x < 1, \alpha, \beta > 0, \quad (1)$$

$$f(x) = \alpha\beta \frac{x^{\alpha-1}}{(1-x)^{\alpha+1}} e^{-\left(\frac{x}{1-x}\right)^\alpha} \left[1 - e^{-\left(\frac{x}{1-x}\right)^\alpha}\right]^{\beta-1}, \quad (2)$$

$$S(x) = 1 - \left[1 - e^{-\left(\frac{x}{1-x}\right)^\alpha}\right]^\beta, \quad (3)$$

$$h(x) = \frac{\alpha\beta \frac{x^{\alpha-1}}{(1-x)^{\alpha+1}} e^{-\left(\frac{x}{1-x}\right)^\alpha} \left[1 - e^{-\left(\frac{x}{1-x}\right)^\alpha}\right]^{\beta-1}}{1 - \left[1 - e^{-\left(\frac{x}{1-x}\right)^\alpha}\right]^\beta}, \quad (4)$$

respectively. The ERK distribution reduces to RK distribution when $\beta = 1$. The hazard function of ERK distribution can be increasing or decreasing depending on the parameters. If $\alpha \in (0,1]$ and $\beta \in (0,1)$, the distribution has decreasing hazard function. If $\alpha > 1$ and $\beta > 1$, it has increasing hazard function.

[11] plotted the skewness and kurtosis values of ERK distribution for different parameter settings. It can be seen from these figures that in the context of skewness, the ERK distribution can be positively or

negatively skewed. In view of kurtosis, the distribution may be short or long tailed. For better understanding the shape of ERK distribution, we draw the pdfs of ERK distribution when $\alpha = 0.5, 3$ and $\beta = 0.5, 1.5, 3, 10$, see Figure 1.

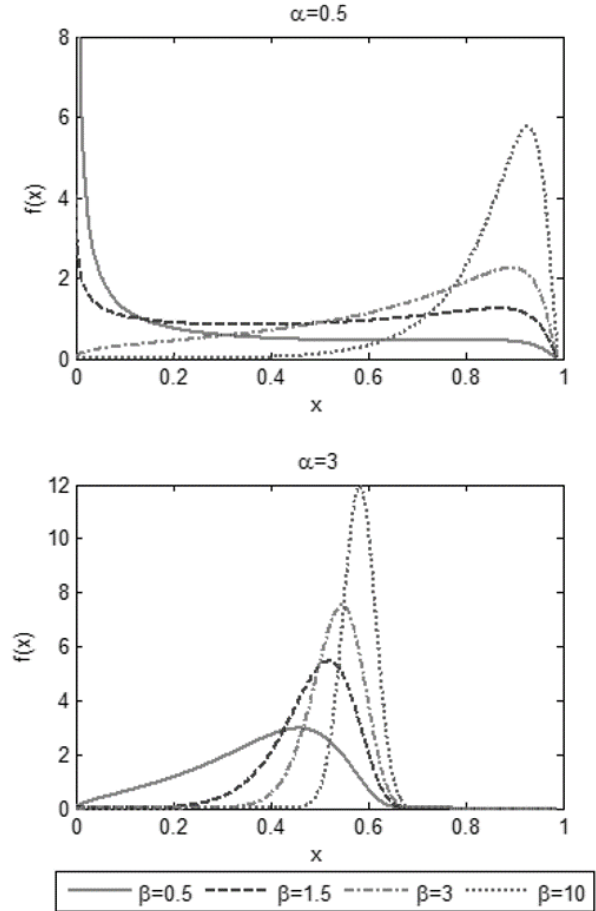


Figure 1. Plots of ERK distribution when $\alpha = 0.5$ and $\alpha = 3$ for different values of β .

2.2. Maximum likelihood estimators

Let $\underline{x} = (x_1, x_2, \dots, x_n)$ be a random sample of size n from ERK distribution with parameters α and β . Then, the log-likelihood function (ℓ) of the observed sample is

$$\begin{aligned} \ell = & n \ln \alpha + n \ln \beta + (\alpha - 1) \sum_{i=1}^n \ln x_i \\ & - (\alpha + 1) \sum_{i=1}^n \ln(1 - x_i) - \sum_{i=1}^n \left(\frac{x_i}{1 - x_i}\right)^\alpha \\ & + (\beta - 1) \sum_{i=1}^n \ln \left(1 - e^{-\left(\frac{x_i}{1 - x_i}\right)^\alpha}\right). \end{aligned} \quad (5)$$

The ML estimators of the parameters α and β are obtained from the following likelihood equations

$$\frac{\partial \ell}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^n \ln(z_i) - \sum_{i=1}^n z_i^\alpha \ln(z_i)$$

$$+(\beta-1) \sum_{i=1}^n \frac{z_i^\alpha \ln(z_i) e^{-z_i^\alpha}}{1-e^{-z_i^\alpha}} = 0, \quad (6)$$

$$\frac{\partial \ell}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^n \ln(1-e^{-z_i^\alpha}) = 0, \quad (7)$$

where, $z_i = \left(\frac{x_i}{1-x_i}\right)$. Obviously, from (7) we obtain

$$\hat{\beta}(\alpha) = -\frac{n}{\sum_{i=1}^n \ln(1-e^{-z_i^\alpha})}. \quad (8)$$

By putting (8) into (6), we get

$$\frac{n}{\alpha} + \sum_{i=1}^n \ln(z_i) - \sum_{i=1}^n z_i^\alpha \ln(z_i) - m = 0. \quad (9)$$

Here,

$$m = \left(\frac{n + \sum_{i=1}^n \ln(1-e^{-z_i^\alpha})}{\sum_{i=1}^n \ln(1-e^{-z_i^\alpha})} \right) \sum_{i=1}^n \frac{z_i^\alpha \ln(z_i) e^{-z_i^\alpha}}{1-e^{-z_i^\alpha}}.$$

Therefore, the ML estimator of α , say $\hat{\alpha}$, can be obtained as a solution of the non-linear equation of the form $h(\alpha) = \alpha$, where

$$h(\alpha) = \frac{n}{\sum_{i=1}^n z_i^\alpha \ln(z_i) + m - \sum_{i=1}^n \ln(z_i)}. \quad (10)$$

It is obvious that (10) cannot be solved explicitly. Therefore, for the ML estimator of α , we resort to iterative methods such as Newton-Raphson. Once, we obtain $\hat{\alpha}$, then $\hat{\beta}$ can be obtained from (8).

2.3. Least squares estimators

Let $x_{(1)} < x_{(2)} < \dots < x_{(n)}$ be the order statistics of a random sample of size n from ERK distribution. The LS estimators of the parameters α and β are obtained by minimizing following equation with respect to the parameters of interest

$$S(\alpha, \beta) = \sum_{i=1}^n \left(F(x_{(i)}) - \frac{i}{n+1} \right)^2. \quad (11)$$

Here, $\frac{i}{n+1}$, ($i = 1, \dots, n$) are the expected values of $F(x_{(i)})$. By incorporating the cdf of ERK distribution given in (1) into (11) and taking the derivative of it with respect to α and β , the LS estimators of the parameters are obtained by solving following nonlinear equations

$$\sum_{i=1}^n \left(F(x_{(i)}, \alpha, \beta) - \frac{i}{n+1} \right) \Delta_1(x_{(i)}, \alpha, \beta) = 0,$$

$$\sum_{i=1}^n \left(F(x_{(i)}, \alpha, \beta) - \frac{i}{n+1} \right) \Delta_2(x_{(i)}, \alpha, \beta) = 0,$$

where $\Delta_1(x, \alpha, \beta)$ and $\Delta_2(x, \alpha, \beta)$ are

$$\Delta_1(x, \alpha, \beta) = \beta \left(\frac{x}{1-x} \right)^\alpha \ln \left(\frac{x}{1-x} \right) e^{-\left(\frac{x}{1-x} \right)^\alpha}$$

$$\left(1 - e^{-\left(\frac{x}{1-x} \right)^\alpha} \right)^{\beta-1}, \quad (12)$$

$$\Delta_2(x, \alpha, \beta) = \ln \left(1 - e^{-\left(\frac{x}{1-x} \right)^\alpha} \right)$$

$$\left(1 - e^{-\left(\frac{x}{1-x} \right)^\alpha} \right)^\beta, \quad (13)$$

respectively.

2.4. Weighted least squares estimators

The WLS estimators of the parameters α and β are obtained by minimizing following equation with respect to the parameters of interest

$$W(\alpha, \beta) = \sum_{i=1}^n w_i \left(F(x_{(i)}) - \frac{i}{n+1} \right)^2. \quad (14)$$

Here,

$w_i = 1/V(F(x_{(i)})) = (n+1)^2(n+2)/i(n-i+1)$, ($i = 1, \dots, n$). By incorporating the cdf of ERK distribution given (1) into (14) and taking the derivative of it with respect to the α and β , the WLS estimators of the parameters are obtained by solving following nonlinear equations

$$\sum_{i=1}^n \frac{1}{i(n-i+1)} \left(F(x_{(i)}, \alpha, \beta) - \frac{i}{n+1} \right)$$

$$\Delta_1(x_{(i)}, \alpha, \beta) = 0,$$

$$\sum_{i=1}^n \frac{1}{i(n-i+1)} \left(F(x_{(i)}, \alpha, \beta) - \frac{i}{n+1} \right)$$

$$\Delta_2(x_{(i)}, \alpha, \beta) = 0,$$

where $\Delta_1(x, \alpha, \beta)$ and $\Delta_2(x, \alpha, \beta)$ are the same as in (12) and (13), respectively.

2.5. Minimum distance methods

In this subsection, we consider three estimation methods for estimating parameters of ERK distribution by minimizing the goodness of fit statistics. These estimators are obtained based on differences between the estimate cdf and the empirical cdf and proposed by [12-13]. In recent years, these methods are very popular in estimation theory. For example, they are used to estimate the parameters of Weibull, Marshall-Olkin extended Lindley, Marshall-Olkin extended exponential and inverse Weibull distributions by [14-17], respectively.

2.5.1. Cramér-von Mises estimators

The CM estimators of the parameters α and β are obtained by minimizing following equation with respect to the parameters of interest

$$C(\alpha, \beta) = \frac{1}{12n} + \sum_{i=1}^n \left(F(x_{(i)}, \alpha, \beta) - \frac{2i-1}{2n} \right)^2. \quad (15)$$

The CM estimators of the parameters are obtained by solving following nonlinear equations

$$\begin{aligned} \sum_{i=1}^n \left(F(x_{(i)}, \alpha, \beta) - \frac{2i-1}{2n} \right) \Delta_1(x_{(i)}, \alpha, \beta) &= 0, \\ \sum_{i=1}^n \left(F(x_{(i)}, \alpha, \beta) - \frac{2i-1}{2n} \right) \Delta_2(x_{(i)}, \alpha, \beta) &= 0, \end{aligned}$$

where $\Delta_1(x, \alpha, \beta)$ and $\Delta_2(x, \alpha, \beta)$ are the same as in (12) and (13), respectively.

2.5.2. Anderson Darling estimators

The AD estimators of the parameters α and β are obtained by minimizing following equation with respect to the parameters of interest

$$A(\alpha, \beta) = -n - \frac{1}{n} \sum_{i=1}^n (2i-1) \log [F(x_{(i)}) (1 - F(x_{(i^*)}))], \quad (16)$$

where $i^* = n - i + 1$.

The AD estimators of the parameters are obtained by solving following nonlinear equations

$$\begin{aligned} \sum_{i=1}^n (2i-1) \left[\frac{\Delta_1(x_{(i)}, \alpha, \beta)}{F(x_{(i)}, \alpha, \beta)} - \frac{\Delta_1(x_{(i^*)}, \alpha, \beta)}{1 - F(x_{(i^*)}, \alpha, \beta)} \right] &= 0, \\ \sum_{i=1}^n (2i-1) \left[\frac{\Delta_2(x_{(i)}, \alpha, \beta)}{F(x_{(i)}, \alpha, \beta)} - \frac{\Delta_2(x_{(i^*)}, \alpha, \beta)}{1 - F(x_{(i^*)}, \alpha, \beta)} \right] &= 0, \end{aligned}$$

where $\Delta_1(x, \alpha, \beta)$ and $\Delta_2(x, \alpha, \beta)$ are the same as in (12) and (13), respectively.

2.5.3. Right-tail Anderson Darling estimators

The RAD estimators of the parameters α and β are obtained by minimizing following equation with respect to the parameters of interest

$$\begin{aligned} R(\alpha, \beta) &= \frac{n}{2} - 2 \sum_{i=1}^n F(x_{(i)}) \\ &\quad - \frac{1}{n} \sum_{i=1}^n (2i-1) \log (1 - F(x_{(i^*)})). \end{aligned} \quad (17)$$

The RAD estimators of the parameters are obtained by solving following nonlinear equations

$$-2 \sum_{i=1}^n \frac{\Delta_1(x_{(i)}, \alpha, \beta)}{F(x_{(i)}, \alpha, \beta)} + \frac{1}{n} \sum_{i=1}^n (2i-1) \frac{\Delta_1(x_{(i^*)}, \alpha, \beta)}{1 - F(x_{(i^*)}, \alpha, \beta)} = 0,$$

$$-2 \sum_{i=1}^n \frac{\Delta_2(x_{(i)}, \alpha, \beta)}{F(x_{(i)}, \alpha, \beta)} + \frac{1}{n} \sum_{i=1}^n (2i-1) \frac{\Delta_2(x_{(i^*)}, \alpha, \beta)}{1 - F(x_{(i^*)}, \alpha, \beta)} = 0,$$

where $\Delta_1(x, \alpha, \beta)$ and $\Delta_2(x, \alpha, \beta)$ are the same as in (12) and (13), respectively.

3. Results

In this section, we give the results of the Monte-Carlo simulation study and the real data application.

3.1. Simulation study

Here, we present the results of Monte-Carlo simulation to compare the performances of the different estimation methods discussed in the previous section. To do this, we compute the means and mean square errors (MSE) of the estimators for each parameter. The sample sizes are taken as $n = 25, 50, 100$ and 500 . In the context of parameter settings, we take $\alpha = 0.5$ and 3 and $\beta = 0.5, 1.5$ and 5 . All the computations are done based on $\llbracket 100,000/n \rrbracket$ Monte-Carlo runs where $\llbracket . \rrbracket$ represents the integer value function. We generate a sample of size n from ERK distribution using inverse cdf method via following expression

$$x = \frac{\eta}{1+\eta},$$

where $\eta = [-\ln(1 - U^{1/\beta})]^{1/\alpha}$ and U is a standard uniform observation.

It should be noticed that the ML, LS, WLS, CV, AD and RAD estimates of the parameters are obtained by using *fminsearch* function in the optimization toolbox of MatlabR2013a software. The results are reported in Tables 1 and 2.

It is observed from Tables 1 and 2 that when $n = 25$ and 50 , the AD and RAD estimates have the smallest bias. On the other hand, the LS and CV estimates have the largest bias. As the sample size increases, all the estimates have negligible bias.

In the context of efficiency, The AD is the most efficient method for α when $n = 25$ according to Tables 1 and 2. However, for the other sample sizes, the ML estimate outperforms other estimates. It is followed by the AD and RAD. These estimates are highly competitive compared to ML in view of the MSEs of $\hat{\alpha}$. It should be stated that the LS and CV estimates do not perform well for all sample sizes.

In terms of the efficiency of $\hat{\beta}$, the ML is the most efficient estimator with the lowest MSE values for all

sample sizes. It is followed by the AD and RAD when $\beta = 0.5$. However, when $\beta = 1.5$ and 5, the RAD estimates do not perform well. In this case, the AD and WLS estimates show the strongest performance

after the ML estimates. It should be noted that the LS and CV estimators of β demonstrate the weakest performance with the highest MSE values for all sample sizes.

Table 1. Simulated mean and MSE values of $\hat{\alpha}$ and $\hat{\beta}$; $\alpha = 0.5, \beta = 0.5, 1.5$ and 5.*

n		ML	LS	WLS	CV	AD	RAD
$\alpha = 0.5, \beta = 0.5$							
25	α	0.5905	0.5750	0.5707	0.6776	0.5488	0.5592
		(0.0600)	(0.1672)	(0.1419)	(0.2990)	(0.0585)	(0.0709)
	β	0.4845	0.5259	0.5178	0.4868	0.5114	0.5118
		(0.0241)	(0.0397)	(0.0335)	(0.0415)	(0.0295)	(0.0309)
50	α	0.5403	0.5311	0.5283	0.5716	0.5197	0.5217
		(0.0138)	(0.0423)	(0.0287)	(0.0665)	(0.0145)	(0.0130)
	β	0.4914	0.5122	0.5068	0.4921	0.5064	0.5065
		(0.0114)	(0.0200)	(0.0159)	(0.0208)	(0.0143)	(0.0144)
100	α	0.5178	0.5114	0.5123	0.5278	0.5104	0.5104
		(0.0048)	(0.0095)	(0.0067)	(0.0115)	(0.0057)	(0.0051)
	β	0.4969	0.5045	0.5017	0.4944	0.5018	0.5020
		(0.0056)	(0.0098)	(0.0076)	(0.0100)	(0.0073)	(0.0072)
500	α	0.5006	0.5017	0.5012	0.5047	0.5005	0.4997
		(0.0007)	(0.0018)	(0.0012)	(0.0019)	(0.0011)	(0.0009)
	β	0.4989	0.4981	0.4982	0.4960	0.4985	0.4986
		(0.0009)	(0.0019)	(0.0014)	(0.0019)	(0.0014)	(0.0013)
$\alpha = 0.5, \beta = 1.5$							
25	α	0.5302	0.5018	0.5063	0.5346	0.5101	0.5146
		(0.0089)	(0.0123)	(0.0106)	(0.0172)	(0.0082)	(0.0085)
	β	1.5338	1.5283	1.5290	1.5358	1.5291	1.5558
		(0.1240)	(0.1354)	(0.1289)	(0.1554)	(0.1249)	(0.1611)
50	α	0.5145	0.5000	0.5039	0.5153	0.5051	0.5073
		(0.0035)	(0.0050)	(0.0041)	(0.0057)	(0.0038)	(0.0037)
	β	1.5244	1.5220	1.5223	1.5251	1.5220	1.5360
		(0.0573)	(0.0665)	(0.0620)	(0.0711)	(0.0613)	(0.0754)
100	α	0.5041	0.4978	0.5002	0.5053	0.5005	0.5013
		(0.0015)	(0.0023)	(0.0019)	(0.0024)	(0.0018)	(0.0018)
	β	1.5156	1.5119	1.5127	1.5131	1.5123	1.5173
		(0.0256)	(0.0300)	(0.0279)	(0.0310)	(0.0277)	(0.0322)
500	α	0.5005	0.4982	0.4994	0.4997	0.4991	0.4996
		(0.0003)	(0.0005)	(0.0003)	(0.0005)	(0.0003)	(0.0003)
	β	1.5004	1.5003	1.5000	1.5005	1.4999	1.5017
		(0.0060)	(0.0072)	(0.0066)	(0.0072)	(0.0066)	(0.0074)
$\alpha = 0.5, \beta = 5$							
25	α	0.5152	0.4958	0.4995	0.5153	0.5031	0.5071
		(0.0031)	(0.0040)	(0.0035)	(0.0046)	(0.0030)	(0.0036)
	β	5.4127	5.1357	5.1790	5.4827	5.2217	5.4498
		(1.8325)	(2.0503)	(1.8843)	(2.8631)	(1.6815)	(3.2998)
50	α	0.5076	0.4981	0.5007	0.5076	0.5018	0.5034
		(0.0014)	(0.0018)	(0.0016)	(0.0020)	(0.0015)	(0.0017)
	β	5.1626	5.0237	5.0575	5.1801	5.0735	5.1510
		(0.6753)	(0.7676)	(0.6877)	(0.8956)	(0.6646)	(0.9679)
100	α	0.5041	0.4990	0.5010	0.5038	0.5011	0.5017
		(0.0007)	(0.0009)	(0.0008)	(0.0009)	(0.0008)	(0.0008)
	β	5.0765	5.0035	5.0304	5.0787	5.0330	5.0653
		(0.2900)	(0.3433)	(0.3053)	(0.3695)	(0.2982)	(0.4092)
500	α	0.4994	0.4990	0.4992	0.4999	0.4992	0.4989
		(0.0001)	(0.0002)	(0.0001)	(0.0002)	(0.0001)	(0.0001)
	β	5.0184	5.0103	5.0150	5.0251	5.0149	5.0122
		(0.0583)	(0.0757)	(0.0652)	(0.0771)	(0.0649)	(0.0863)

*MSEs of the estimates are given in the bracket.

Table 2. Simulated mean and MSE values of $\hat{\alpha}$ and $\hat{\beta}$; $\alpha=3, \beta=0.5, 1.5$ and 5 .*

n		ML	LS	WLS	CV	AD	RAD
$\alpha=3, \beta=0.5$							
25	α	3.5553 (2.2127)	3.4967 (6.8165)	3.4695 (6.0934)	4.1034 (11.7191)	3.3002 (2.3803)	3.3815 (3.4381)
	β	0.4856 (0.0236)	0.5275 (0.0398)	0.5191 (0.0334)	0.4889 (0.0413)	0.5131 (0.0289)	0.5135 (0.0304)
	α	3.2195 (0.4823)	3.1696 (1.4699)	3.1569 (0.9933)	3.4027 (2.1393)	3.1105 (0.5398)	3.1177 (0.4739)
	β	0.4913 (0.0107)	0.5106 (0.0200)	0.5056 (0.0158)	0.4905 (0.0207)	0.5049 (0.0143)	0.5053 (0.0144)
100	α	3.1486 (0.1897)	3.1149 (0.4591)	3.1122 (0.3020)	3.2168 (0.5598)	3.0950 (0.2366)	3.0974 (0.1964)
	β	0.4897 (0.0055)	0.5001 (0.0106)	0.4970 (0.0081)	0.4899 (0.0109)	0.4974 (0.0077)	0.4976 (0.0076)
	α	3.0168 (0.0261)	3.0041 (0.0698)	3.0099 (0.0435)	3.0218 (0.0718)	3.0054 (0.0409)	3.0046 (0.0336)
	β	0.5020 (0.0010)	0.5045 (0.0020)	0.5033 (0.0015)	0.5024 (0.0020)	0.5036 (0.0015)	0.5036 (0.0014)
$\alpha=3, \beta=1.5$							
25	α	3.1931 (0.3322)	3.0186 (0.4230)	3.0423 (0.3576)	3.2169 (0.5883)	3.0743 (0.3067)	3.1008 (0.3113)
	β	1.5347 (0.1209)	1.5282 (0.1315)	1.5290 (0.1255)	1.5356 (0.1509)	1.5294 (0.1221)	1.5564 (0.1575)
	α	3.0909 (0.1275)	2.9950 (0.1826)	3.0199 (0.1492)	3.0870 (0.2086)	3.0260 (0.1330)	3.0465 (0.1324)
	β	1.5039 (0.0531)	1.5083 (0.0618)	1.5068 (0.0572)	1.5111 (0.0661)	1.5062 (0.0562)	1.5221 (0.0692)
100	α	3.0501 (0.0543)	3.0115 (0.0820)	3.0266 (0.0651)	3.0566 (0.0890)	3.0255 (0.0605)	3.0328 (0.0603)
	β	1.5134 (0.0253)	1.5129 (0.0308)	1.5132 (0.0282)	1.5142 (0.0319)	1.5125 (0.0278)	1.5195 (0.0337)
	α	3.0122 (0.0100)	3.0037 (0.0144)	3.0096 (0.0111)	3.0126 (0.0147)	3.0075 (0.0111)	3.0120 (0.0108)
	β	1.4924 (0.0045)	1.4951 (0.0057)	1.4942 (0.0051)	1.4953 (0.0057)	1.4942 (0.0051)	1.4966 (0.0060)
$\alpha=3, \beta=5$							
25	α	3.0916 (0.1130)	2.9757 (0.1427)	2.9948 (0.1231)	3.0931 (0.1657)	3.0197 (0.1099)	3.0426 (0.1280)
	β	5.3580 (1.6426)	5.0884 (1.8748)	5.1165 (1.6370)	5.4314 (2.6461)	5.1715 (1.4839)	5.3735 (2.7755)
	α	3.0533 (0.0512)	2.9971 (0.0690)	3.0118 (0.0576)	3.0547 (0.0752)	3.0184 (0.0541)	3.0327 (0.0611)
	β	5.2098 (0.7635)	5.0923 (0.8816)	5.1173 (0.7882)	5.2531 (1.0478)	5.1310 (0.7542)	5.2354 (1.1626)
100	α	3.0132 (0.0228)	2.9870 (0.0328)	2.9954 (0.0266)	3.0155 (0.0336)	2.9969 (0.0258)	3.0006 (0.0281)
	β	5.0759 (0.2813)	5.0126 (0.3478)	5.0319 (0.3089)	5.0885 (0.3758)	5.0355 (0.2990)	5.0695 (0.4303)
	α	2.9994 (0.0051)	2.9956 (0.0072)	2.9981 (0.0058)	3.0013 (0.0072)	2.9969 (0.0058)	2.9971 (0.0059)
	β	4.9940 (0.0582)	4.9849 (0.0727)	4.9910 (0.0623)	4.9996 (0.0733)	4.9878 (0.0613)	4.9922 (0.0817)

*MSEs of the estimates are given in the bracket.

3.2. Real data application

Now, we model the relief times (in hours) for 50 arthritic patients with ERK distribution. This data set was first given by [18] to make an implementation of the estimation of the parameters of Burr XII distribution. The data set is shown as follows:

Table 3. The relief times data set

0.70	0.84	0.58	0.50	0.55	0.82	0.59	0.71	0.72	0.61
0.62	0.49	0.54	0.36	0.36	0.71	0.35	0.64	0.85	0.55
0.59	0.29	0.75	0.46	0.46	0.60	0.60	0.36	0.52	0.68
0.80	0.55	0.84	0.34	0.34	0.70	0.49	0.56	0.71	0.61
0.57	0.73	0.75	0.44	0.44	0.81	0.80	0.87	0.29	0.50

Before starting the analyses, we fit the ERK model to the data set. To do this, we draw the ERK q-q plot in Figure 2. It is observed from Figure 2 that ERK distribution provides very good fit to model the relief times data set.

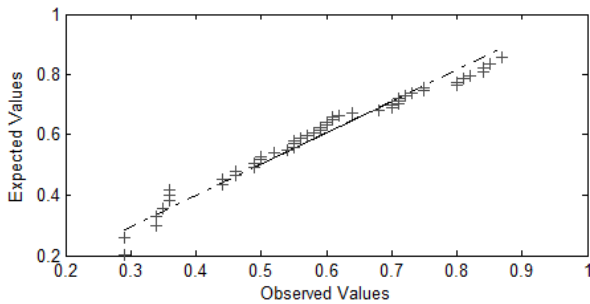


Figure 2. The ERK q-q plot for relief times data set.

Then, we obtain the ML, LS, WLS, CM, AD and RAD estimates of the parameters. Furthermore, to determine the most efficient estimation methods, we use the model selection criteria. They are Akaike information criterion (AIC), Bayesian information criterion (BCI), corrected ACI (ACIc) and Hannan-Quinn criterion (HQC). It should be noted that the smallest values of these criteria represent the best fit. The results are given in Table 4.

Table 4. Parameter estimates and model selection criteria values for relief time data

	$\hat{\alpha}$	$\hat{\beta}$	ACI	BCI	ACIc	HQC
ML	0.89	2.89	-38.58	-26.93	-38.32	-37.12
LS	0.87	2.57	-37.85	-26.20	-37.59	-36.39
WLS	0.88	2.71	-38.35	-26.70	-38.09	-36.89
CV	0.89	2.61	-38.06	-26.41	-37.80	-36.60
AD	0.87	2.67	-38.18	-26.53	-37.92	-36.72
RAD	0.83	2.51	-36.88	-25.24	-36.63	-35.43

It is obvious from Table 4 that the ML estimates have the smallest model selection criteria values. In other words, the ERK model based on the ML estimates is the most appropriate model among the others. Furthermore, they are followed by the WLS and AD estimates. For illustration of these results, we draw the histogram of the data set with fitted pdfs in Figure 3. According to Figure 3 that while the pdfs based on ML, WLS and AD estimates provides good fit, the pdfs based on LS, CV and RAD remain incapable to model the data set.

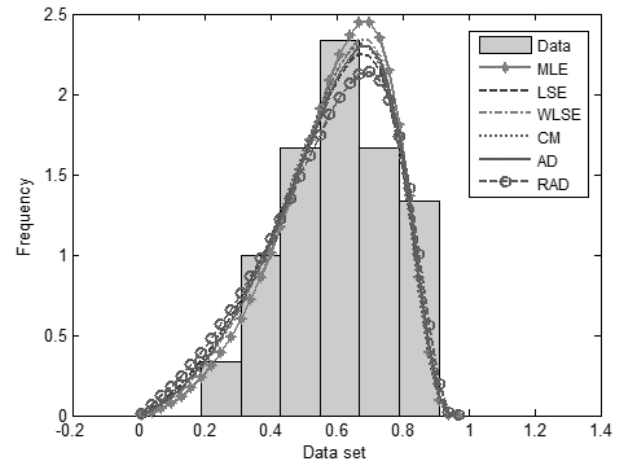


Figure 3. Histogram of relief time data with fitted pdfs.

4. Discussion and Conclusion

In this paper, we consider the estimation of the parameters of ERK distribution by using classical methods ML, LS and WLS, and minimum distance methods CM, AD and RAD. The performances of the estimators are compared via Monte-Carlo simulation study. It is concluded from the simulation study that the ML estimators demonstrate the best performances among them. Furthermore, AD and WLS estimators work quite well. It should be stated that the minimum distance method AD is highly competitive method compared to ML. However, LS and CV estimators do not perform well. These results are supported with the real data example.

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