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Investigation of Determinants of Fibonacci-Hessenberg-Lorentz Matrices and Special Number Sequences

İbrahim Gökcan[1,*], Ali Hikmet Değer[2]

[1]*Faculty of Arts and Sciences, Artvin Çoruh University, 08000, Artvin, Türkiye.*
[2]*Department of Mathematics, Faculty of Science, Karadeniz Technical University, 61000, Trabzon, Türkiye.*

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Abstract. The research aims to construct a new type of matrix called the Fibonacci-Hessenberg-Lorentz matrix by multiplying Fibonacci-Hessenberg matrices with Lorentz matrix multiplication. The study will start by examining the properties of Hessenberg and tridiagonal matrices and then focus on developing the Fibonacci-Hessenberg matrix using Fibonacci sequences. By multiplication it with a Lorentz matrix multiplication, the resulting matrix, the Fibonacci-Hessenberg-Lorentz matrix, will be analyzed to obtain special number sequences through its determinants for $n \geq 1$. The primary objective is to explore whether the determinants of these matrices can generate new or known number sequences, where the elements are expressed as functions of the matrix parameters. Furthermore, the research will attempt to generalize these sequences of using Fibonacci numbers to establish a generalized formula for their terms. Ultimately, the goal is to derive a mathematical representation that connects the characteristics of the newly defined matrices to well-known special sequences in mathematics.

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1. Introduction

In recent decades, Fibonacci-type sequences and their generalizations have been widely studied due to their rich algebraic structures and diverse applications. Classical results link the determinants of Hessenberg and tridiagonal matrices to Fibonacci and Lucas numbers, providing elegant matrix-based representations of well-known sequences. Numerous studies have attempted to investigate number sequences; especially [1, 3, 4, 11, 12, 19, 20, 23, 29, 30] can be given as references. In [14], generalization of Lucas numbers is demonstrated and a connection is attained between k-Lucas numbers and Fibonacci numbers. In addition, generalized k-Fibonacci and Lucas numbers are analyzed, Binet formulas are investigated for these sequences in [28]. In [22], a closed formula are accessed for Horadam polynomials and polynomial sequences by using tridiagonal determinants. A closed formula giving the elements of Horadam sequence are achieved by tridiagonal matrices in [15]. The formulas related to (p, q, r)-tribonacci polynomials and generalized tribonacci sequences are presented by using Hessenberg matrices in [16]. Such studies emphasize the usefulness of matrix methods in deriving closed forms, Binet-type formulas, and recurrence relations. An extensive body of research has been dedicated to investigating the algebraic and spectral properties of various structured matrices. In [13], an efficient algorithm was proposed for reducing (p, q)-pentadiagonal matrices to tridiagonal form using permutation matrices. The study not only provided explicit forms of these permutation matrices but also analysed the

*Corresponding Author
Email addresses: gokcan@artvin.edu.tr (İ. Gökcan), ahikmetd@ktu.edu.tr (A.H. Değer)

sparsity pattern of the resulting tridiagonal matrices in detail. Building upon classical matrix types, [17] introduced generalized versions of the classical min and max matrices, referred to as r-min and r-max matrices, showing that they reduce to their classical counterparts when $r = 1$. The study also thoroughly examined their determinants, inverses, norms, decompositions, and Hadamard inverses. In a similar vein, [24] explored a new classes of matrices called geometric r-Frank matrices, as an extension of both the classical and generalized Frank matrices. This work investigated their algebraic properties, factorizations, determinant expressions, and various matrix norms. Furthermore, [7] focused on pseudo k-tridiagonal matrices, presenting results concerning their determinants, spectra, and inverses. In [9], a new family of symmetric Toeplitz matrices was constructed using elements from specific Horadam sequences. The study provided explicit expressions of their determinants, inverses, and matrix factorizations, and demonstrated that these matrices are not always singular. It was also shown that their inverses exhibit a pentadiagonal structure. In [25], a generalized Frank matrix, namely the r-Frank matrix, was introduced, encompassing several other generalized forms of Frank matrices. The study addressed its algebraic structure, determinant, inverse, various norms, and matrix factorizations, while also examining several linear algebraic properties of its inverse. Additionally, [8] presented two new families of matrices offering general expressions for their determinants and inverse, which also cover some special cases of Toeplitz and related matrices. Lastly, [26] investigated the r-Hankel and r-Toeplitz matrices composed of geometric sequences, analysing their algebraic and spectral characteristics, including determinants, inverses, generalized inverses, and spectral norms.

Parallel to these developments, Lorentz matrix multiplication has attracted interest as a less conventional but powerful operation in matrix theory, with applications ranging from functional analysis to computational mathematics. However, its combination with Fibonacci-Hessenberg structures has not been sufficiently explored. The purpose of this paper is to introduce Fibonacci-Hessenberg-Lorentz matrices, obtained by applying Lorentz matrix multiplication to Fibonacci-Hessenberg matrices, and to analyse their determinants. Our main goal is to determine whether these determinants generate new or known number sequences and to establish explicit recurrence relations. In doing so, we position our results within the broader literature on generalized sequences and highlight new structural connections arising from Lorentz multiplication.

Hessenberg and Tridiagonal Matrices and Its Determinants. The term α_{ij} is used here to refer to an element in a matrix. Several definitions of Hessenberg matrix have been proposed. First, Hessenberg matrix is defined as matrix $n \times n$ such that $\alpha_{j,j+1} \neq 0$ and $\alpha_{ij} = 0$ for $j > i + 1$ but not lower triangular. Second, main diagonal and its lower triangular matrix are compose of elements different from 0, but upper triangular matrix compose of elements 0. The following matrix is a good illustration of Hessenberg matrix:

$$H_n = \begin{pmatrix} h_{1,1} & h_{1,2} & \cdots & 0 & 0 \\ a_{2,1} & a_{2,2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & 0 & 0 \\ h_{n-1,1} & h_{n-1,2} & \cdots & h_{n-1,n-1} & h_{n-1,n} \\ h_{n,1} & h_{n,2} & \cdots & h_{n,n-1} & h_{n,n} \end{pmatrix}.$$

Assume that the sequence consisting of the determinants of the Hessenberg matrix H_n is $\{|H_n|, n \geqslant 1\}$. For instance, for $n = 1$, $H_n = h_{1,1}$. For $n \geqslant 2$, the general formula giving determinants is:

$$|H_n| = h_{n,n}|H_{n-1}| + \sum_{r=1}^{n-1} \left[(-1)^{n-r} h_{n,r} \prod_{j=r}^{n-1} h_{j,j+1} |H_{r-1}| \right].$$

There are multiple definitions of a tridiagonal matrix. One of these can be given as if the elements of different from 0 only on main diagonal, upper diagonal and lower diagonal, matrix is defined as tridiagonal matrix. Mathematically, the tridiagonal matrix can be defined as $\alpha_{ij} = 0$ for $j > i + 1$ and $i > j + 1$, $\alpha_{ii} \neq 0$ and $\alpha_{j,j+1} \neq 0$ for in other situations. The following matrix illustrates a tridiagonal matrix clearly.

$$T_n = \begin{pmatrix} t_{1,1} & t_{1,2} & \cdots & 0 & 0 \\ t_{2,1} & t_{2,2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & \cdots & t_{n-1,n-1} & t_{n-1,n} \\ 0 & 0 & \cdots & t_{n,n-1} & t_{n,n} \end{pmatrix}.$$

However, the set of Hessenberg matrices cover the set of tridiagonal matrices. Therefore, obtaining determinant of a tridiagonal matrix can be used above equation. For some special values, the determinants of the Hessenberg and Tridiagonal matrices were associated with the general terms of the Fibonacci number sequence. This finding broadly supports the work of other studies in this area linking Hessenberg and Tridiagonal matrices with Fibonacci number sequence. A considerable amount of literature has been published on Hessenberg matrices whose determinants are related to Fibonacci numbers. Here, [2,5,6,21,27] can be given as a reference to some studies.
Here, some matrix examples and findings that have been extensively examined in the literature are given. In [27], it is shown that the determinant of the matrix, whose elements on the main diagonal are 1, elements on the upper and lower diagonals are i, and other elements are 0, is F_{n+1} for $n \in \mathbb{N}$.
This matrix has been shown as follows with matrix A_n:

$$A_n = \begin{pmatrix} 1 & i & \cdots & 0 & 0 \\ i & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & \cdots & 1 & i \\ 0 & 0 & \cdots & i & 1 \end{pmatrix}.$$

In [2], b_{ij} is defined as an element in matrix B_n. In addition, assume that the elements of main diagonal are 2 but $b_{nn} = 1$, the elements of upper diagonal and lower diagonal matrix are 1. Then, matrix B_n can be demonstrated in the following:

$$B_n = \begin{pmatrix} 2 & 1 & \cdots & 0 & 0 \\ 1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & 0 & 0 \\ 1 & 1 & \cdots & 2 & 1 \\ 1 & 1 & \cdots & 1 & 1 \end{pmatrix}.$$

Matrix B_n can be entitled upper Hessenberg matrix because of the elements of upper diagonal matrix are 0 and main diagonal and elements of lower diagonal matrix are different from 0. One interesting finding relevant to is the determinants of matrix B_n are $|B_1| = 1, |B_2| = 1$ and $|B_n| = F_n$ for respectively $1, 2$ and n.
In [27], the determinants of Hessenberg matrices whose elements are complex numbers are associated with Fibonacci sequences. In [5], the authors examined the determinants of tridiagonal matrices whose elements are real numbers and associated determinants with Fibonacci sequences.
In previous studies, the first and second type Fibonacci numbers, $F_{n-1}t + F_{n-2}$ and $F_{n-1} + F_{n-2}t$ respectively, are found by giving a variable t in the elements $(1, 1)$ or (n, n) of some Hessenberg and tridiagonal matrices.

Definition 1.1. Assume that t is a real or complex number and n is an integer. The terms of the first and second type Fibonacci numbers, $F_{n-1}t + F_{n-2}$ and $F_{n-1} + F_{n-2}t$ respectively are used in its broadest sense to refer to all the determinant of the matrices created depending on t and n. In the present study, the first and second type Fibonacci numbers is defined as (t, n)-Fibonacci.

Definition 1.2. Assume that H_n is a Hessenberg matrix and t is an integer providing $m > 0$ such that $\forall n \geqslant m$. Hessenberg H_n matrix is named as Fibonacci-Hessenberg matrix if determinants of Hessenberg H_n matrix are (t, n)-Fibonacci numbers.

Here, a well-known examples of some Fibonacci-Hessenberg matrices and sequences created by its determinants with (t, n)-Fibonacci numbers are presented for $n \geqslant 1$.
First, a classic example of Hessenberg matrix is $C_{n,t}$ that its determinants are shown Fibonacci numbers:

$$C_{n,t} = \begin{pmatrix} 2 & 1 & \cdots & 0 & 0 \\ 1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & 0 & 0 \\ 1 & 1 & \cdots & 2 & 1 \\ 1 & 1 & \cdots & 1 & t+1 \end{pmatrix}.$$

For $n \geqslant 1$, determinants of matrix $C_{n,t}$ are indicated as follows:
$|C_{1,t}| = t+1, |C_{2,t}| = 2t+1, |C_{3,t}| = 3t+2, \cdots, |C_{n,t}| = F_{n+1}t + F_n$. $C_{n,t}$ is the first type (t,n)-Fibonacci number. Further analysis shows that the sequences $\{0,-1,-1,\cdots,-F_{n-1},\cdots\}$, $\{1,1,2,\cdots,F_n,\cdots\}$,$\{2,3,5,\cdots,F_{n+2},\cdots\}$ and $\{3,5,8,\cdots,F_{n+3},\cdots\}$ are achieved for respectively $t = -1, 0, 1$ and 2.
Second, matrix $D_{n,t}$ is obtained by replacing which element are 1 in the upper diagonal of the matrix $C_{n,t}$ with -1. In the following, the matrix $D_{n,t}$ and its determinants are demonstrated:

$$D_{n,t} = \begin{pmatrix} 2 & -1 & \cdots & 0 & 0 \\ 1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & 0 & 0 \\ 1 & 1 & \cdots & 2 & -1 \\ 1 & 1 & \cdots & 1 & t+1 \end{pmatrix}.$$

$|D_{1,t}| = t+1, |D_{2,t}| = 2t+3, |D_{3,t}| = 3t+5, \cdots, |D_{n,t}| = F_{2n-1}t + F_{2n}$. The results obtained from analysis of determinants of matrix $D_{n,t}$ are $\{0,1,2,\cdots,F_{2n-2},\cdots\}$, $\{1,3,5,\cdots,F_{2n},\cdots\}$,$\{2,5,8,\cdots,F_{2n+1},\cdots\}$ and $\{3,7,11,\cdots,L_{2n},\cdots\}$ for respectively $t = -1, 0, 1$ and 2.
Third, the other Fibonacci-Hessenberg matrix is matrix $E_{n,t}$ achieving by replacing which element is 2 in the $(1,1)$ of the matrix $D_{n,t}$ with 1. The matrix $E_{n,t}$ can be illustrated and generalized determinants can be presented as follows:

$$E_{n,t} = \begin{pmatrix} 1 & -1 & \cdots & 0 & 0 \\ 1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & 0 & 0 \\ 1 & 1 & \cdots & 2 & -1 \\ 1 & 1 & \cdots & 1 & t+1 \end{pmatrix}$$

$|E_{1,t}| = t+1, |E_{2,t}| = 2t+3, |E_{3,t}| = 3t+5, \cdots, |E_{n,t}| = F_{2n-2}t + F_{2n-1}$. From operations of determinant we can see that sequences $\{0,1,2,\cdots,F_{2n-3},\cdots\}$, $\{1,3,5,\cdots,F_{2n-1},\cdots\}$, $\{2,5,8,\cdots,F_{2n},\cdots\}$ and $\{3,7,11,\cdots,L_{2n-1},\cdots\}$ are achieved for respectively $t = -1, 0, 1$ and 2.
Fourth, Fibonacci-Hessenberg matrix $F_{n,t}$ is found by replacing which element is 1 in the $(1,1)$ of the matrix $E_{n,t}$ with 2. In addition, lower diagonal matrix are compose of elements of -1 and 1 respectively. A matrix representing the matrix $F_{n,t}$ is given below:

$$F_{n,t} = \begin{pmatrix} 2 & -1 & \cdots & 0 & 0 \\ -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & 0 & 0 \\ -1 & 1 & \cdots & 2 & -1 \\ 1 & -1 & \cdots & -1 & t+1 \end{pmatrix}$$

For $n \geqslant 1$, determinants of matrix $F_{n,t}$ are $|F_{1,t}| = t+1, |F_{2,t}| = 2t+1, |F_{3,t}| = 3t+2, \cdots, |F_{n,t}| = F_n + tF_{n+1}$. As can be seen from the operations of determinant, sequences $\{0,-1,-1,\cdots,-F_{n-1},\cdots\}$, $\{1,1,2,\cdots,F_n,\cdots\}$, $\{2,3,5,\cdots,F_{n+2},\cdots\}$ and $\{3,5,8,\cdots,F_{n+3},\cdots\}$ for respectively $t = -1, 0, 1$ and 2.
Fifth, Fibonacci-Hessenberg matrix $G_{n,t}$ is accessed by replacing which element is 2 in the $(1,1)$ of the matrix $F_{n,t}$ with 1. The represantation of matrix $G_{n,t}$ and generalized determinants have given in the following.

$$G_{n,t} = \begin{pmatrix} 1 & -1 & \cdots & 0 & 0 \\ -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & 0 & 0 \\ -1 & 1 & \cdots & 2 & -1 \\ 1 & -1 & \cdots & -1 & t+1 \end{pmatrix}$$

$|G_{1,t}| = t+1, |G_{2,t}| = 2t+1, |G_{3,t}| = 3t+2, \cdots, |G_{n,t}| = F_{n-2} + tF_{n-1}$. If $t = -1, 0, 1$ and 2 are taken, sequences $\{0,-1,-1,\cdots,-F_{n-3},\cdots\}$,$\{1,1,2,\cdots,F_{n-2},\cdots\}$, $\{2,3,5,\cdots,F_n,\cdots\}$, $\{3,5,8,\cdots,F_{n+1},\cdots\}$ are respectively obtained.
Sixth, Fibonacci-Hessenberg matrix $H_{n,t}$ is defined by replacing which element are -1 in the upper diagonal of the

matrix $G_{n,t}$ with 1. The matrix $H_{n,t}$ and its generalized determinants can be given as follows:

$$H_{n,t}=\begin{pmatrix}1 & 1 & \cdots & 0 & 0\\ -1 & 2 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & 0 & 0\\ -1 & 1 & \cdots & 2 & 1\\ 1 & -1 & \cdots & -1 & t+1\end{pmatrix}.$$

$|H_{1,t}| = t+1, |H_{2,t}| = 2t+3, |H_{3,t}| = 5t+8, \cdots, |H_{n,t}| = F_{2n-1}+tF_{2n-2}$. The following can be presented about the sequences attained with the determinants of the matrix $H_{n,t}$: sequences $\{0,1,3,\cdots,F_{2n-3},\cdots\}$, $\{1,3,8,\cdots,F_{2n-1},\cdots\}$, $\{2,5,13,\cdots,F_{2n},\cdots\}$ and $\{3,7,18,\cdots,L_{2n-1},\cdots\}$ for respectively $t=-1,0,1$ and 2.
Seventh, Fibonacci-Hessenberg matrix $K_{n,t}$ is realized by replacing which element is 1 in $(1,1)$ of the matrix $H_{n,t}$ with 2. The matrix $K_{n,t}$ can be visualized and its generalized determinants can be given as follows:

$$K_{n,t}=\begin{pmatrix}2 & 1 & \cdots & 0 & 0\\ -1 & 2 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & 0 & 0\\ -1 & 1 & \cdots & 2 & 1\\ 1 & -1 & \cdots & -1 & t+1\end{pmatrix}.$$

$|K_{1,t}| = t+1, |K_{2,t}| = 2t+3, |K_{3,t}| = 5t+8, \cdots, |K_{n,t}| = F_{2n}+tF_{2n-1}$. The sequences derived from the evaluation of the determinants of the matrix $K_{n,t}$ can be summarized as below: sequences $\{0,1,3,\cdots,F_{2n-2},\cdots\}$,$\{1,3,8,\cdots,F_{2n},\cdots\}$, $\{2,5,13,\cdots,F_{2n+1},\cdots\}$ and $\{3,7,18,\cdots,L_{2n},\cdots\}$ for respectively $t=-1,0,1$ and 2.
The transitions from one to another in the Fibonacci-Hessenberg matrices given just above can be visualized as follows. In the next matrix, the changes with respect to the first matrix are depicted in red.

$$C_{n,t}=\begin{pmatrix}2 & 1 & \cdots & 0 & 0\\ 1 & 2 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & 0 & 0\\ 1 & 1 & \cdots & 2 & 1\\ 1 & 1 & \cdots & 1 & t+1\end{pmatrix}\longrightarrow D_{n,t}=\begin{pmatrix}2 & -1 & \cdots & 0 & 0\\ 1 & 2 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & 0 & 0\\ 1 & 1 & \cdots & 2 & -1\\ 1 & 1 & \cdots & 1 & t+1\end{pmatrix}\longrightarrow$$

$$E_{n,t}=\begin{pmatrix}1 & -1 & \cdots & 0 & 0\\ 1 & 2 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & 0 & 0\\ 1 & 1 & \cdots & 2 & -1\\ 1 & 1 & \cdots & 1 & t+1\end{pmatrix}\longrightarrow F_{n,t}=\begin{pmatrix}2 & -1 & \cdots & 0 & 0\\ -1 & 2 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & 0 & 0\\ -1 & 1 & \cdots & 2 & -1\\ 1 & -1 & \cdots & -1 & t+1\end{pmatrix}\longrightarrow$$

$$G_{n,t}=\begin{pmatrix}1 & -1 & \cdots & 0 & 0\\ -1 & 2 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & 0 & 0\\ -1 & 1 & \cdots & 2 & -1\\ 1 & -1 & \cdots & -1 & t+1\end{pmatrix}\longrightarrow H_{n,t}=\begin{pmatrix}1 & 1 & \cdots & 0 & 0\\ -1 & 2 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & 0 & 0\\ -1 & 1 & \cdots & 2 & 1\\ 1 & -1 & \cdots & -1 & t+1\end{pmatrix}\longrightarrow$$

$$K_{n,t}=\begin{pmatrix}2 & 1 & \cdots & 0 & 0\\ -1 & 2 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & 0 & 0\\ -1 & 1 & \cdots & 2 & 1\\ 1 & -1 & \cdots & -1 & t+1\end{pmatrix}.$$

The sequences generated with the determinants of the Fibonacci-Hessenberg matrices mentioned previously and the recurrence relations are listed in the below table 1.

Table 1. Number sequences generated by the determinants of Fibonacci-Hessenberg matrices $C_{n,t}, D_{n,t}, E_{n,t}, F_{n,t}, G_{n,t}, H_{n,t}, K_{n,t}$, for different values of t.Each row corresponds to distinct a matrix family, while the columns illustrate the first few sequence elements and the associated recurrence relation.

Matrix $C_{n,t}$	$n = 1$	$n = 2$	$n = 3$	$\cdots$	n
$t = -1$	0	-1	-1	$\cdots$	$-F_{n-1}$
$t = 0$	1	1	2	$\cdots$	F_n
$t = 1$	2	3	5	$\cdots$	F_{n+2}
$t = 2$	3	5	8	$\cdots$	F_{n+3}
t	$t + 1$	$2t + 1$	$3t + 2$	$\cdots$	$F_{n+1}t + F_n$
Matrix $D_{n,t}$	**$n = 1$**	**$n = 2$**	**$n = 3$**	$\cdots$	**n**
$t = -1$	0	1	2	$\cdots$	F_{2n-2}
$t = 0$	1	3	5	$\cdots$	F_{2n}
$t = 1$	2	5	8	$\cdots$	F_{2n+1}
$t = 2$	3	7	11	$\cdots$	L_{2n}
t	$t + 1$	$2t + 3$	$3t + 5$	$\cdots$	$F_{2n-1}t + F_{2n}$
Matrix $E_{n,t}$	**$n = 1$**	**$n = 2$**	**$n = 3$**	$\cdots$	**n**
$t = -1$	0	1	2	$\cdots$	F_{2n-3}
$t = 0$	1	3	5	$\cdots$	F_{2n-1}
$t = 1$	2	5	8	$\cdots$	F_{2n}
$t = 2$	3	7	11	$\cdots$	L_{2n-1}
t	$t + 1$	$2t + 3$	$3t + 5$	$\cdots$	$F_{2n-2}t + F_{2n-1}$
Matrix $F_{n,t}$	**$n = 1$**	**$n = 2$**	**$n = 3$**	$\cdots$	**n**
$t = -1$	0	-1	-1	$\cdots$	$-F_{n-1}$
$t = 0$	1	1	2	$\cdots$	F_n
$t = 1$	2	3	5	$\cdots$	F_{n+2}
$t = 2$	3	5	8	$\cdots$	F_{n+3}
t	$t + 1$	$2t + 1$	$3t + 2$	$\cdots$	$F_n + tF_{n+1}$
Matrix $G_{n,t}$	**$n = 1$**	**$n = 2$**	**$n = 3$**	$\cdots$	**n**
$t = -1$	0	-1	-1	$\cdots$	$-F_{n-3}$
$t = 0$	1	1	2	$\cdots$	F_{n-2}
$t = 1$	2	3	5	$\cdots$	F_n
$t = 2$	3	5	8	$\cdots$	F_{n+1}
t	$t + 1$	$2t + 1$	$3t + 2$	$\cdots$	$F_{n-2} + tF_{n-1}$
Matrix $H_{n,t}$	**$n = 1$**	**$n = 2$**	**$n = 3$**	$\cdots$	**n**
$t = -1$	0	1	3	$\cdots$	F_{2n-3}
$t = 0$	1	3	8	$\cdots$	F_{2n-1}
$t = 1$	2	5	13	$\cdots$	F_{2n}
$t = 2$	3	7	18	$\cdots$	L_{2n-1}
t	$t + 1$	$2t + 3$	$5t + 8$	$\cdots$	$F_{2n-1} + tF_{2n-2}$
Matrix $K_{n,t}$	**$n = 1$**	**$n = 2$**	**$n = 3$**	$\cdots$	**n**
$t = -1$	0	1	3	$\cdots$	F_{2n-2}
$t = 0$	1	3	8	$\cdots$	F_{2n}
$t = 1$	2	5	13	$\cdots$	F_{2n+1}
$t = 2$	3	7	18	$\cdots$	L_{2n}
t	$t + 1$	$2t + 3$	$5t + 8$	$\cdots$	$F_{2n} + tF_{2n-1}$

Note that F_n denotes the n^{th} Fibonacci number, L_n denotes the n^{th} Lucas number. The parameter t indicates that the modified entry in the matrix definition. For each matrix type, both explicit sequences and closed recurrence relations are listed.

$C^i_{n,t}, D^i_{n,t}$ and $E^i_{n,t}$ Fibonacci-Hessenberg Matrices. In previous studies, in Fibonacci-Hessenberg matrices $C_{n,t}, D_{n,t}$ and $E_{n,t}$, by substituting the i^{th} column by column 1, the matrices whose determinants are t-Fibonacci numbers are produced. These matrices are indicated by $C^i_{n,t}, D^i_{n,t}$ and $E^i_{n,t}$. Given the sequence $\{|C^i_{n,t}|, n \geqslant 1\}$ for the matrix $C^i_{n,t}$, the recurrence relation is then derived as $|C^i_{n,t}| = tF_{n-i} + F_{n-i-1}, n \geqslant i \geqslant 1$. Therefore, the recurrence relation that yields the sequence of determinants of the matrix $C_{n,t}$ is $|C^i_{n,t}| = t + \sum_{i=1}^{n} |C^i_{n,t}|$. Likewise, for a matrix $D^i_{n,t}$, the recurrence relation which gives the sequence $\{|D^i_{n,t}|, n \geqslant 1\}$ is acquired as $|D^i_{n,t}| = F_{2(n-i)+1} + tF_{2(n-i)}, n \geqslant i \geqslant 1$. In a similar way, the recurrence relation $|D^i_{n,t}| = F_{2(n-i)+1} + tF_{2(n-i)}, n \geqslant i \geqslant 1$ for the matrix $D^i_{n,t}$ gives the sequence $\{|D^i_{n,t}|, n \geqslant 1\}$. Consequently, it is the recurrence relation $|D_{n,t}| = t + \sum_{i=1}^{n} |D^i_{n,t}|$ that provides the sequence of determinants of the matrix $D_{n,t}$. In addition, the recurrence relation to give the sequence $\{|E^i_{n,t}|, n \geqslant 1\}$ is determined as $|E^i_{n,t}| = tF_{n-i} + F_{n-i+1}, n \geqslant i \geqslant 2$.Furthermore, the recurrence relation $2|E_{n,t}| = (t+1) + \sum_{i=1}^{n} |E^i_{n,t}|, n \geqslant 2$ satisfies the sequence of determinants of the matrix $E_{n,t}$.
The t-Fibonacci numbers of Fibonacci-Hessenberg matrices $C^i_{n,t}, D^i_{n,t}$ and $E^i_{n,t}$ can be seen in the table 2 for $i = 1, 2$ and 3.

Table 2. Generalized t-Fibonacci numbers arising from Fibonacci-Hessenberg matrices $C^i_{n,t}, D^i_{n,t}$ and $E^i_{n,t}$ for $i = 1, 2$ and 3

	Matrix $C^i_{n,t}$	Matrix $D^i_{n,t}$	Matrix $E^i_{n,t}$
$i = 1$	$tF_{n-1} + F_{n-2}$	$F_{2n-1} + tF_{2n-2}$	$tF_{n-1} + F_n$
$i = 2$	$tF_{n-2} + F_{n-3}$	$F_{2n-3} + tF_{2n-4}$	$tF_{n-2} + F_{n-1}$
$i = 3$	$tF_{n-3} + F_{n-4}$	$F_{2n-5} + tF_{2n-6}$	$tF_{n-3} + F_{n-2}$

Lorentz Matrix Multiplication. This part highlights the key theoretical concepts which related to obtaining of Lorentz matrices . A considerable amount of literature has been published on Lorentz matrices. These studies have been widely investigated by researchers.
Suppose that the set of matrices with m rows and n columns consisting of real numbers is denoted by R^m_n. The matrix set R^m_n is a real vector space with respect to addition and scalar multiplication. [18] can be referenced for Lorentz matrix multiplication. Now, suppose there are two vectors $x = (x_1, x_2, \cdots, x_n)$ and $y = (y_1, y_2, \cdots, y_n) \in R^n_n$. The inner product between the two vectors is described as given below:

$$\langle x, y\rangle = -x_1 y_1 + \sum_{i=2}^{n} x_i y_i.$$

The set of the Lorentz matrices to apply the Lorentz inner product is designated by L^n_n.

Lorentz Matrix Multiplication and its Properties. Assume that $A = [a_{ij}] \in R^m_n$ and $B = [b_{jk}] \in R^n_p$. In addition, assume that $\cdot_L$ is a Lorentz matrix multiplication between matrices A and B. Then,

$$A \cdot_L B = \left[-a_{i1} b_{1k} + \sum_{j=2}^{n} a_{ij} b_{jk}\right] \tag{1.1}$$

is determined. That is, $A \cdot_L B$ is a $m \times p$ type matrix. More over, every element in $A \cdot_L B$ is a Lorentz inner product. Also let A_i be the i^{th} row in A and B^j the j^{th} column in B. Thus, the $(i, j)^{th}$ element of $A \cdot_L B$ is $\langle A_i, B^j\rangle_L$. In this way, $A \cdot_L B$ can be identified as the below:

$$A \cdot_L B = \begin{pmatrix} \langle A_1, B^1\rangle_L & \cdots & \langle A_1, B^j\rangle_L \\ \vdots & \ddots & \vdots \\ \langle A_j, B^1\rangle_L & \cdots & \langle A_j, B^j\rangle_L \end{pmatrix}.$$

Theorem 1.3 ([10]). *The below properties related to the Lorentz matrix product are ensured:*

i. $\forall A \in L_n^m, B \in L_p^n$ *for* $A \cdot_L (B \cdot_L C) = (A \cdot_L B) \cdot_L C$
ii. $\forall A \in L_n^m, B, C \in L_p^n$ *for* $A \cdot_L (B + C) = A \cdot_L B + A \cdot_L C$
iii. $\forall A, B \in L_n^m, C \in L_p^n$ *for* $(A + B) \cdot_L C = A \cdot_L C + B \cdot_L C$
iv. $\forall k \in R, A \in L_n^m, B \in L_p^n$ *for* $k(A \cdot_L B) = (kA) \cdot_L B = A \cdot_L (kB)$.

With respect to Lorentz matrix multiplication, assuming that the unit matrix of type $n \times n$ is labeled I_n^n. That is,

$$I_n^n = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \cdot_L \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Definition 1.4 ([10]). A is called invertible and denoted by A^{-1} if for $A \in L_n^n$ if there exists a matrix $B \in L_n^n$ of type $n \times n$ satisfying the equation $A \cdot_L B = B \cdot_L A = I_n^n$.

Definition 1.5 ([10]). The transpose of the matrix $A \in L_n^m$ is represented by A^T and is then given by $A^T = \left[a_{ji}\right] \in L_m^n$.

Definition 1.6 ([10]). $A \in L_n^m$ is called L-ortogonal if $A^{-1} = A^T$.

Definition 1.7 ([10]). The L-determinant of the matrix $A = \left[a_{ij}\right] \in R_n^n$ is identified by $\det A$ and given by $\det A = \sum_{\sigma \in S_n} s(\sigma)\, a_{s(1)} a_{s(2)2} \cdots a_{s(n)n}$. Note that S_n is a set containing all permutations of the set $\{1, 2, \cdots, n\}$. Furthermore, $s(\sigma)$ is the sign of the permutations of σ.

Theorem 1.8 ([10]). *For every* $A, B \in L_n^n$, $det\,(A \cdot_L B) = -\det A \cdot \det B$ *is provided.*

2. Results

The methods for obtaining a matrix have varied somewhat across this research area. Different methods have been proposed to achieve a matrix. The use of Lorentz inner product is one of the most well known methods for attaining a matrix. In this part, the matrix to which Lorentz matrix multiplication is applied is specifically denoted by A_L. Suppose that for $A = \left[a_{ij}\right] \in R_n^n$ and $B = \left[b_{jk}\right] \in R_n^n$, $A \cdot_L B = AB_L$ is defined. Here, the matrix AB_L is called Fibonacci-Hessenberg-Lorentz matrix. By the Theorem 1.8, $\det AB_L = -\det A \cdot \det B$ for $A, B \in L_n^n$. By applying Lorentz matrix multiplication to the $n \times n$ type matrices discussed in the literature, it allows us to reach completely new matrices. Via the determinants of the related matrices, new sequences of numbers associated with Fibonacci numbers have been included in the literature.

Conclusion 2.1. *Suppose $C_{n,t}$ and $D_{n,t}$ are Fibonacci-Hessenberg matrices and $\cdot_L$ is the Lorentz matrix product. The determinant of the matrix derived under Lorentz matrix multiplication of the $C_{n,t}$ and $D_{n,t}$ matrices for $n \geqslant 1$ yields the sequence as follows:*

$$|CD_{n,t}| = \{-t^2 - 2t - 1, -4t^2 - 8t - 3, -15t^2 - 34t - 16, \cdots, -F_{n+1}F_{2n-1}t^2 - (F_{n+1}F_{2n} + F_nF_{2n-1})\,t - F_nF_{2n}, \cdots\}.$$

Proof. For $n = 1$, $C_{1,t} = \left(t + 1\right)$ and $D_{1,t} = \left(t + 1\right)$ are provided. Then, applying Lorentz multiplication gives

$$C_{1,t} \cdot_L D_{1,t} = CD_{1,t} = \left(-t^2 - 2t - 1\right).$$

By Theorem 1.8,

$$\det CD_{1,t} = -\det C_{1,t} \cdot \det D_{1,t} = -(t + 1) \cdot (t + 1) = -t^2 - 2t - 1$$

is found. For $n = 2$, $C_{2,t} = \begin{pmatrix} 2 & 1 \\ 1 & t+1 \end{pmatrix}$ and $D_{2,t} = \begin{pmatrix} 2 & -1 \\ 1 & t+1 \end{pmatrix}$ are satisfied. Using Lorentz multiplication formula (1.1), we compute first row $(-2 \cdot 2 + 1 \cdot 1, -2 \cdot (-1) + 1(t + 1)) = (-3, t + 3)$, second row $(-1(1)(2) + (t + 1)(1), -(1)(-1) + (t + 1)(t + 1)) = (t - 1, t^2 + 2t + 2)$. So,

$$CD_{2,t} = C_{2,t} \cdot_L D_{2,t} = \begin{pmatrix} 2 & 1 \\ 1 & t+1 \end{pmatrix} \cdot_L \begin{pmatrix} 2 & -1 \\ 1 & t+1 \end{pmatrix} = \begin{pmatrix} -3 & t+3 \\ t-1 & t^2+2t+2 \end{pmatrix}.$$

The determinant of $CD_{2,t}$,

$$\det CD_{2,t} = (-3)(t^2 + 2t + 2) - (t + 3)(t - 1) = -4t^2 - 8t - 3$$

is attained. On the other hand, $\det C_{2,t} = 2(t+1) - 1 = 2t+1$, $\det D_{2,t} = 2(t+1)+1 = 2t+3$. Thus, $-\det C_{2,t} \cdot \det D_{2,t} = -(2t+1)\cdot(2t+3) = -4t^2 - 8t - 3$, which matches. The matrices for $n = 3$ are equal to $C_{3,t} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & t+1 \end{pmatrix}$ and $D_{3,t} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & t+1 \end{pmatrix}$. Applying the Lorentz matrix multiplication to these resulting matrices produces

$$CD_{3,t} = C_{3,t} \cdot_L D_{3,t} = \begin{pmatrix} -3 & 4 & -1 \\ 1 & 6 & t-1 \\ t & t+4 & t^2+2t \end{pmatrix}.$$

The determinant given in the Theorem 1.8 provides

$$\det CD_{3,t} = -\det C_{3,t} \cdot \det D_{3,t} = -(3t+2)\cdot(5t+8) = -15t^2 - 34t - 16.$$

It is determined as $C_{n,t} = \begin{pmatrix} 2 & 1 & \cdots & 0 & 0 \\ 1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & 0 & 0 \\ 1 & 1 & \cdots & 2 & 1 \\ 1 & 1 & \cdots & 1 & t+1 \end{pmatrix}$ and $D_{n,t} = \begin{pmatrix} 2 & -1 & \cdots & 0 & 0 \\ 1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & 0 & 0 \\ 1 & 1 & \cdots & 2 & -1 \\ 1 & 1 & \cdots & 1 & t+1 \end{pmatrix}$ for the general n term. Then, the determinants are known $\det C_{n,t} = F_{n+1}t + F_n$, $\det D_{n,t} = F_{2n-1}t + F_{2n}$. By Theorem 1.8 on Lorentz determinants,

$$\det CD_{n,t} = -\det C_{n,t} \cdot \det D_{n,t} = -(F_{n+1}t + F_n)\cdot(F_{2n-1}t + F_{2n}) = -F_{n+1}F_{2n-1}t^2 - (F_{n+1}F_{2n} + F_nF_{2n-1})\,t - F_nF_{2n}$$

is derived. Consequently, the below sequences of numbers are arrived at:

$$|CD_{n,t}| = \{-t^2 - 2t - 1, -4t^2 - 8t - 3, -15t^2 - 34t - 16, \cdots, -F_{n+1}F_{2n-1}t^2 - (F_{n+1}F_{2n} + F_nF_{2n-1})\,t - F_nF_{2n}, \cdots\}.$$

□

Conclusion 2.2. *Let $E_{n,t}$ and $F_{n,t}$ be the Fibonacci-Hessenberg matrices and $\cdot_L$ be the Lorentz matrix product. For $n \geqslant 1$, the determinant of the matrix $EF_{n,t}$ generated under the Lorentz matrix multiplication of $E_{n,t}$ and $F_{n,t}$ gives the next sequence:*

$$|EF_{n,t}| = \{-t^2 - 2t - 1, -4t^2 - 8t - 3, -15t^2 - 34t - 16, \cdots, -F_{2n-2}F_{n+1}t^2 - (F_{2n-2}F_n + F_{n+1}F_{2n-1}) - F_{2n-1}F_n, \cdots\cdots\}.$$

Proof. Note that for $n = 1$, $E_{1,t} = \begin{pmatrix} t+1 \end{pmatrix}$ and $F_{1,t} = \begin{pmatrix} t+1 \end{pmatrix}$ are achieved. After that, Lorentz multiplication gives:

$$E_{1,t} \cdot_L F_{1,t} = EF_{1,t} = \begin{pmatrix} -t^2 - 2t - 1 \end{pmatrix}$$

Hence, applying the determinant operation according to the Lorentz matrix multiplication, it is clear that

$$\det EF_{1,t} = -\det E_{1,t} \cdot \det F_{1,t} = -(t+1)\cdot(t+1) = -t^2 - 2t - 1.$$

In the following, related Lorentz matrix and its determinant value are presented for matrices $E_{2,t} = \begin{pmatrix} 2 & -1 \\ 1 & t+1 \end{pmatrix}$ and $F_{2,t} = \begin{pmatrix} 2 & -1 \\ -1 & t+1 \end{pmatrix}$. Now compute $E_{2,t} \cdot_L F_{2,t} = EF_{2,t}$. First row $(-2\cdot 2 + (-1)(-1), -2(-1) + (-1)(t+1)) = (-3, -t+1)$. Second row $(-(1)(2) + (t+1)(-1), -(1)(-1) + (t+1)(t+1)) = (-t-3, t^2+2t+2)$. So,

$$EF_{2,t} = \begin{pmatrix} -3 & -t+1 \\ -t-3 & t^2+2t+2 \end{pmatrix}.$$

Determinant of $EF_{2,t}$

$$\det EF_{2,t} = -\det E_{2,t} \cdot \det F_{2,t} = (-3)(t^2 + 2t + 2) - (-t + 1)(-t - 3) = -4t^2 - 8t - 3.$$

On the other hand, for $\det E_{2,t} = 2(t+1)+1 = 2t+3$, $\det F_{2,t} = 2(t+1)-1 = 2t+1$, $-\det E_{2,t} \det F_{2,t} = -(2t+3)(2t+1) = -4t^2 - 8t - 3$. By applying similar operations, $E_{3,t} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & t+1 \end{pmatrix}$ and $F_{3,t} = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 2 & -1 \\ 1 & -1 & t+1 \end{pmatrix}$ matrices are reached for $n = 3$. Subsequently, the corresponding Lorentz matrix is equal to

$$E_{3,t} \cdot_L F_{3,t} = EF_{3,t} = \begin{pmatrix} -3 & 0 & 1 \\ -5 & 6 & -t-3 \\ t-2 & -t+2 & t^2+2t \end{pmatrix}.$$

On the other hand, the determinant for matrix $EF_{3,t}$ is

$$\det EF_{3,t} = -\det E_{3,t} \cdot \det F_{3,t} = -(5t + 8) \cdot (3t + 2) = -15t^2 - 34t - 16.$$

It is known that $\det E_{n,t} = F_{2n-2}t + F_{2n-1}$,$\det F_{n,t} = F_n + tF_{n+1}$. By Lorentz determinant property in Theorem 1.8, a Lorentz matrix is also generated from the matrices $E_{n,t} = \begin{pmatrix} 1 & -1 & \cdots & 0 & 0 \\ 1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & 0 & 0 \\ 1 & 1 & \cdots & 2 & -1 \\ 1 & 1 & \cdots & 1 & t+1 \end{pmatrix}$ and $F_{n,t} = \begin{pmatrix} 2 & -1 & \cdots & 0 & 0 \\ -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & 0 & 0 \\ -1 & 1 & \cdots & 2 & -1 \\ 1 & -1 & \cdots & -1 & t+1 \end{pmatrix}$ of type $n \times n$ and its determinant is then assigned as

$$\det EF_{n,t} = -\det E_{n,t} \cdot \det F_{n,t} = -(F_{2n-2}t + F_{2n-1}) \cdot (F_n + tF_{n+1}) = -F_{2n-2}F_{n+1}t^2 - (F_{2n-2}F_n + F_{n+1}F_{2n-1})\,t - F_{2n-1}F_n.$$

As a result, the following sequences of numbers are produced:

$$|EF_{n,t}| = \{-t^2 - 2t - 1, -4t^2 - 8t - 3, -15t^2 - 34t - 16, \cdots, -F_{2n-2}F_{n+1}t^2 - (F_{2n-2}F_n + F_{n+1}F_{2n-1})\,t - F_{2n-1}F_n, \cdots\}.$$

□

Conclusion 2.3. *For $n \geqslant 1$, the determinant of the matrix $GH_{n,t}$ formed by the Lorentz matrix multiplication of $G_{n,t}$ and $H_{n,t}$ produces the subsequent sequence:*

$$|GH_{n,t}| = \{-t^2 - 2t - 1, -4t^2 - 8t - 3, -15t^2 - 34t - 16, \cdots, -F_{2n-2}F_{n+1}t^2 - (F_{2n-2}F_n + F_{n+1}F_{2n-1})\,t - F_{2n-1}F_n, \cdots\}$$

Proof. The matrices $G_{n,t}$ and $H_{n,t}$ are given as $G_{1,t} = \begin{pmatrix} t+1 \end{pmatrix}$, $H_{1,t} = \begin{pmatrix} t+1 \end{pmatrix}$ for $n = 1$. Lorentz multiplication gives:

$$GH_{1,t} = G_{1,t} \cdot_L H_{1,t} = (-(t + 1)(t + 1)) = (-t^2 - 2t - 1).$$

So, $\det GH_{1,t} = -\det G_{1,t} \cdot \det H_{1,t} = -(t + 1) \cdot (t + 1) = -t^2 - 2t - 1$. For $n = 2$, $G_{2,t} = \begin{pmatrix} 2 & -1 \\ -1 & t+1 \end{pmatrix}$ and $H_{2,t} = \begin{pmatrix} 2 & 1 \\ -1 & t+1 \end{pmatrix}$ are obtained. Under Lorentz multiplication, first row $(-2 \cdot 2 + (-1)(-1), -2(1) + (-1)(t + 1)) = (-3, -t - 3)$ and second row $(-(-1)(2) + (t + 1)(-1), -(-1)(1) + (t + 1)(t + 1)) = (t - 1, t^2 + 2t + 2)$. So,

$$GH_{2,t} = \begin{pmatrix} -3 & -t-3 \\ t-1 & t^2+2t+2 \end{pmatrix}$$

is found. The determinant of $GH_{2,t}$

$$\det GH_{2,t} = (-3)(t^2 + 2t + 2) - (-t - 3)(t - 1) = -4t^2 - 8t - 3.$$

Meanwhile, $\det G_{2,t} = 2(t + 1) - 1 = 2t + 1$, $\det H_{2,t} = 2(t + 1) + 1 = 2t + 3$. So, $-\det G_{2,t} \det H_{2,t} = -(2t + 1)(2t + 3) = -4t^2 - 8t - 3$. For $n = 3$, $G_{3,t} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 1 & -1 & t+1 \end{pmatrix}$ and $H_{3,t} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & -1 & t+1 \end{pmatrix}$ are determined. Here, Lorentz multiplication gives:

$$GH_{3,t} = \begin{pmatrix} -3 & -4 & -1 \\ -1 & 6 & -t+1 \\ t & -t-4 & t^2+2t \end{pmatrix}.$$

The determinant of $GH_{3,t}$ can be obtained as $\det GH_{3,t} = -\det G_{3,t} \cdot \det H_{3,t} = -(3t+2)\cdot(5t+8) = -15t^2 - 34t - 16$.

For general term n, $G_{n,t} = \begin{pmatrix} 1 & -1 & \cdots & 0 & 0 \\ -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & 0 & 0 \\ -1 & 1 & \cdots & 2 & -1 \\ 1 & -1 & \cdots & -1 & t+1 \end{pmatrix}$ and $H_{n,t} = \begin{pmatrix} 1 & 1 & \cdots & 0 & 0 \\ -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & 0 & 0 \\ -1 & 1 & \cdots & 2 & 1 \\ 1 & -1 & \cdots & -1 & t+1 \end{pmatrix}$ are satisfied. It is known that $\det G_{n,t} = F_{n-2} + tF_{n-1}$ and $\det H_{n,t} = F_{2n-1} + tF_{2n-2}$. By equation $GH_{n,t} = G_{n,t} \cdot_L H_{n,t}$ and Theorem 1.8,

$$\det GH_{n,t} = -\det G_{n,t} \cdot \det H_{n,t} = -(F_{n-2} + tF_{n-1}) \cdot (F_{2n-1} + tF_{2n-2}) =$$
$$-F_{n-1}F_{2n-2}t^2 - (F_{n-2}F_{2n-2} + F_{n-1}F_{2n-1})\,t - F_{n-2}F_{2n-1}$$

is found. So, the following sequences of numbers are derived:

$$|GH_{n,t}| = \{-t^2 - 2t - 1, -4t^2 - 8t - 3, -15t^2 - 34t - 16, \cdots, -F_{2n-2}F_{n-1}t^2 - (F_{2n-2}F_n + F_{n+1}F_{2n-1})\,t - F_{2n-1}F_n, \cdots\}.$$

□

The table of sequences with determinants of Fibonacci-Hessenberg-Lorentz matrices $|CD_{n,t}|$, $|EF_{n,t}|$ and $|GH_{n,t}|$ for different values of n and t can be illustrated as table 3.

Table 3. Sequences generated by the determinants of Fibonacci-Hessenberg-Lorentz matrices $|CD_{n,t}|, |EF_{n,t}|, |GH_{n,t}|$ under Lorentz multiplication for selected values of t. Each entry illustrates the first few terms of the sequence together with the closed-form determinant identity

	Element				
$\lvert CD_{n,t}\rvert$	1^{th}	2^{nd}	3^{rd}	$\cdots$	n^{th}
$t=-1$	0	1	3	$\cdots$	$-F_{n+1}F_{2n-1} + (F_{n+1}F_{2n} + F_nF_{2n-1}) - F_nF_{2n}$
$t=0$	-1	-3	-16	$\cdots$	$-F_nF_2n$
$t=1$	-4	-15	-65	$\cdots$	$-F_{n+1}F_{2n-1} - (F_{n+1}F_{2n} + F_nF_{2n-1}) - F_nF_{2n}$
$t=2$	-9	-35	-144	$\cdots$	$-4F_{n+1}F_{2n-1} - 2\,(F_{n+1}F_{2n} + F_nF_{2n-1}) - F_nF_{2n}$

	Element				
$\lvert EF_{n,t}\rvert$	1^{th}	2^{nd}	3^{rd}	$\cdots$	n^{th}
$t=-1$	0	1	3	$\cdots$	$-F_{2n-2}F_{n+1} + (F_{2n-2}F_n + F_{n+1}F_{2n-1}) - F_{2n-1}F_n$
$t=0$	-1	-3	-16	$\cdots$	$-F_{2n-1}F_n$
$t=1$	-2	-15	-65	$\cdots$	$-F_{2n-2}F_{n+1} - (F_{2n-2}F_n + F_{n+1}F_{2n-1}) - F_{2n-1}F_n$
$t=2$	-9	-35	-144	$\cdots$	$-4F_{2n-2}F_{n+1} - 2\,(F_{2n-2}F_n + F_{n+1}F_{2n-1}) - F_{2n-1}F_n$

	Element				
$\lvert GH_{n,t}\rvert$	1^{th}	2^{nd}	3^{rd}	$\cdots$	n^{th}
$t=-1$	0	1	3	$\cdots$	$-F_{n-1}F_{2n-2} + (F_{n-2}F_{2n-2} + F_{n-1}F_{2n-1}) - F_{n-2}F_{2n-1}$
$t=0$	-1	-3	-16	$\cdots$	$-F_{n-2}F_{2n-1}$
$t=1$	-2	-15	-65	$\cdots$	$-F_{n-1}F_{2n-2}t^2 - (F_{n-2}F_{2n-2} + F_{n-1}F_{2n-1}) - F_{n-2}F_{2n-1}$
$t=2$	-9	-35	-144	$\cdots$	$-F_{n-1}F_{2n-2}t^2 - 2\,(F_{n-2}F_{2n-2} + F_{n-1}F_{2n-1}) - F_{n-2}F_{2n-1}$

3. Conclusion

In this study, we have defined and investigated Fibonacci-Hessenberg-Lorentz matrices by combining the well-known Fibonacci-Hessenberg structures with Lorentz matrix multiplication. Through detailed determinant analysis, we derived new families of number sequences and established their recurrence relations. These sequences enrich the existing theory of special sequences by creating direct links between classical Fibonacci-type numbers and matrix products under Lorentz operations.

The mathematical significance of these results lies in extending the matrix-based representation of special sequences. The new families we obtain not only parallel existing Fibonacci-related structures but also provide a framework for generating additional generalizations. Importantly, the determinant identities developed here demonstrate how Lorentz multiplication can serve as a novel tool in algebraic sequence construction.

Beyond their intrinsic theoretical value, the results may also have potential applications in combinatorial enumeration problems, algebraic structures defined by recurrence relations, and even mathematical physics, where Lorentzian structures naturally appear.

Future research could extend the present approach in several directions. For example, one may apply Lorentz matrix multiplication to matrices associated with Tribonacci, Pell, or Horadam sequences, investigate generalized Hessenberg forms under other inner product structures, or explore computational aspects of these determinant sequences in algorithmic number theory.

Overall, the present paper demonstrates that the intersection of Fibonacci theory and Lorentz multiplication opens a promising research avenue that connects discrete mathematics, algebra, and applied analysis.

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Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

These authors contributed equally to this work.

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