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New Results on Negative-Indexed Pell Numbers via Matrix Methods

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ABSTRACT

In this study, we investigate the Pell and Pell–Lucas numbers sequences and construct matrices whose elements are defined using negative indices of these sequences through Binet's formula. Identities involving negative-indexed Pell and Pell–Lucas numbers are derived using the determinants and traces of these matrices. Furthermore, we examine various powers of the constructed matrices and discuss related elementary matrix operations.

Keywords: *Pell numbers, Pell-Lucas numbers, matrix theory.*

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Introduction

Special number sequences have been intensely analyzed in the related literature and lots of new identities generated by special number sequences have been contributed to the number theory. Many methods are available to derive identities with respect to special sequences of numbers, and the matrix method is one of the most notable. The matrices that have a crucial role in analyzing systems of linear equations are widely used in technology production. Thus, the identities derived via matrices by using special sequences of numbers are significant scientific facts. The Pell and Pell-Lucas number sequences are among the important numerical sequences and contribute to various fields such as coding theory and information theory [1, 2]. Pell and Pell-Lucas number sequences have been investigated by many authors in terms of matrices [3-7]. In [8], the authors give new matrices related to Pell and Pell-Lucas numbers by using the recurrence relations of the continued fractions which are produced from suborbital graphs. Also, in [9], the authors put forward new identities related to Pell and Pell-Lucas numbers by using these matrices. Pell and Pell-Lucas sequences are used in many fields and some of them can be presented as follows. They provide integer solutions to Pell's equation. In addition, these sequences have applications in number theory, especially in Diophantine equations and continued fractions. Also, Pell and Pell-Lucas sequences have applications in algorithmic solutions, especially in cryptography and factorization. Moreover, they have common properties with Fibonacci-like sequences and are in the center of algebraic number theory. The negative indexed Pell numbers are investigated by many authors in various fields in number theory [10-13]. In [14], negative indexed Lucas, Fibonacci and complex Fibonacci numbers are studied to obtain some identities related to sums of finite products of these numbers. Also, these numbers are taken as linear combination of Pell polynomials. Moreover, similar results are achieved for Fibonacci polynomials and Pell numbers using Chebyshev polynomials. In [15], identities were derived using the elements of the negative-indexed k-generalized Fibonacci sequence, and these negative-indexed identities were used in solution of the zero-multicipity problem. In [16], the authors examine generalized Tetranacci quaternions and give information about some of their properties.

In this paper, firstly the matrices related to the negative indexed Pell and Pell-Lucas numbers are introduced. Then, the matrices of the negative indexed Pell and Pell-Lucas numbers are provided and these matrices are examined to produce new identities of these sequences. Consequently, matrices with negative indices are obtained. The negative-indexed matrices derived in this study have applications across various fields, which can be outlined as follows. Matrices with negative indices are an important tool in engineering, especially for modeling time-delayed, past data-dependent, or sequential computation-based systems. In this type of matrix, negative indices indicate the system's previous states or past samples of the signal, thus enabling a more accurate mathematical modeling of dynamic behavior. In signal processing and digital filtering, negative-indexed matrices are used to describe filter coefficients, time shifts, and signal transformation in a compact matrix form. In addition, in numerical simulations, particularly in finite difference methods and discrete-time analyses, matrices with negative indices are used to represent the previous steps of differential equations, allowing the calculations to be performed. Negative-indexed matrices are used intensively in engineering optimization problems and prediction algorithms where previous data have been utilized.

The Pell and Pell-Lucas number sequences are given by the recurrence relations $P_n = 2P_{n-1} + P_{n-2}$, $n \geq 2$, $n \in \mathbb{Z}^+$ by initial conditions $P_0 = 0$ and $P_1 = 1$, $Q_n = 2Q_{n-1} + Q_{n-2}$, $n \geq 2$, $n \in \mathbb{Z}^+$and initial conditions are $Q_0 = 1$ and $Q_1 = 1$, respectively. Here, P_n is the n^{th}Pell number and $Q_n$is the n^{th} Pell-Lucas number. The Binet-like formulas for P_n and Q_n are

$$P_n = \frac{\gamma^n - \delta^n}{\gamma - \delta} \text{ and } Q_n = \frac{\gamma^n + \delta^n}{2}, n \geq 2, n \in \mathbb{Z}^+,$$

where $\gamma = 1 + \sqrt{2}$ and $\delta = 1 - \sqrt{2}$.
The negative indexed Pell numbers are given in the relation

$$P_{-n} = (-1)^{n+1} P_n \tag{1}$$

where P_{-n} is the n^{th} the negative indexed Pell number. This means that the Pell sequence is symmetric in a way similar to the Fibonacci numbers, with alternating signs for negative indices. Similarly, for Pell-Lucas numbers the following equation give the negative indexed Pell-Lucas number,

$$Q_{-n} = (-1)^{n+1} Q_n \tag{2}$$

where Q_{-n} is the n^{th} the negative indexed Pell-Lucas number.
Some identities are related to Pell and Pell-Pell Lucas numbers are given as follow;

mma 1. [10] If P_n is the n^{th} Pell number and Q_n is the n^{th} Pell-Lucas number, then the following equations hold:

$$P_n + P_{n-1} = Q_n,$$
$$Q_n + Q_{n-1} = 2P_n,$$
$$P_{n+1} + P_{n-1} = 2Q_n,$$
$$P_{2n} = 2P_n Q_n,$$
$$Q_{n+1} + Q_{n-1} = 4P_n,$$

where $n \geq 1, n \in \mathbb{Z}^+$.

Theorem 1. [10] Let take P_n is the n^{th} Pell number and $L = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$, the following equation is given

$$L^n = \begin{pmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{pmatrix}, \tag{3}$$

where $n \geq 1$.

Theorem 2. [8] If P_n is the n^{th} Pell number, then

$$\begin{pmatrix} 0 & -1 \\ 1 & -6 \end{pmatrix}^n = \begin{pmatrix} \frac{(-1)^{n-1}}{2} P_{2n-2} & \frac{(-1)^n}{2} P_{2n} \\ \frac{(-1)^{n+1}}{2} P_{2n} & \frac{(-1)^n}{2} P_{2n+2} \end{pmatrix} = V_n \tag{4}$$

where $n \in \mathbb{Z}^+$.

Theorem 3. [8] If Q_n is the n^{th} Pell-Lucas number, then

$$\begin{pmatrix} -6 & 2 \\ -2 & 6 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & -6 \end{pmatrix}^n =$$

$$\begin{pmatrix} 2(-1)^{n-1} Q_{2n-2} & 2(-1)^n Q_{2n} \\ 2(-1)^{n+1} Q_{2n} & 2(-1)^n Q_{2n+2} \end{pmatrix} = W_n, \tag{5}$$

where $n \in \mathbb{Z}^+$.

The objective of this study is to analyze the negative indices of the Pell and Pell-Lucas number sequences by using matrix operations and to present new results and structures. The motivation of this study is to focus on negative-indexed results by investigating negative-indexed matrices, which are less studied than positive-indexed matrices. Therefore, new results will be provided related to negative-indexed numbers. Using the Binet formulas for the Pell and Pell-Lucas numbers given above, the matrices were expressed with negative indices and subsequently obtained in their power forms. The resulting identities are presented through basic operations on these negative-indexed matrices. As a result, the results provided in this paper can contribute to the solving problems in many areas such as optimization problems and prediction algorithms.

Main Results

In this section, we introduce the new matrices related to the negative indexed Pell and Pell-Lucas numbers. Then, we produce new identities by using these matrices.

Theorem 4. For $n \geq 1$ and $L = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$,

$$L^n = \begin{pmatrix} (-1)^{n+2} P_{-(n+1)} & (-1)^{n+1} P_{-n} \\ (-1)^{n+1} P_{-n} & (-1)^n P_{-(n-1)} \end{pmatrix} \tag{6}$$

is given.

Proof. For $1 \leq i, j \leq 2$, let the elements of the matrix L^n be h_{ij}. If the elements are rewritten by identity $P_{-n} = (-1)^{n+1} P_n$, $h_{11} = P_{n+1} = (-1)^{n+2} P_{-(n+1)}$, $h_{12} = P_n = (-1)^{n+1} P_{-n}$, $h_{21} = P_n = (-1)^{n+1} P_{-n}$ and $h_{22} = P_{n-1} = (-1)^n P_{-(n-1)}$ are obtained. So, the matrix which produces the negative Pell numbers is given as;

$$L^n = \begin{pmatrix} (-1)^{n+2}P_{-(n+1)} & (-1)^{n+1}P_{-n} \\ (-1)^{n+1}P_{-n} & (-1)^{n}P_{-(n-1)} \end{pmatrix}.$$

The validity of this matrix is proved using the method of induction. The matrix L^n can be represented as

$$\begin{pmatrix} (-1)^{n}P_{-(n+1)} & (-1)^{n+1}P_{-n} \\ (-1)^{n+1}P_{-n} & (-1)^{n}P_{-(n-1)} \end{pmatrix}.$$

This matrix is true for $n = 1$. Here, the negative –indexed Pell elements are $P_0 = 0, P_{-1} = 1$ and $P_{-2} = -2$.

$$L^1 = \begin{pmatrix} (-1)^{3}P_{-2} & (-1)^{2}P_{-1} \\ (-1)^{2}P_{-1} & (-1)^{1}P_{0} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$$

The matrix is assumed to be true for $n = k$.

$$L^k = \begin{pmatrix} (-1)^{k}P_{-(k+1)} & (-1)^{k+1}P_{-k} \\ (-1)^{k+1}P_{-k} & (-1)^{k}P_{-(k-1)} \end{pmatrix}$$

Then, the matrix is true for $n = k + 1$.

$$L^{k+1} = \begin{pmatrix} (-1)^{k}P_{-(k+1)} & (-1)^{k+1}P_{-k} \\ (-1)^{k+1}P_{-k} & (-1)^{k}P_{-(k-1)} \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$$

The elements h_{ij} for the matrix L^{k+1} are found as follows:

$$h_{11} = (-1)^k P_{-(k+1)}2 + (-1)^{k+1}P_{-k}1 = (-1)^{k+1}\left(P_{-k} - 2P_{-(k+1)}\right) = (-1)^{k+1}P_{-(k+2)}$$
$$h_{12} = (-1)^k P_{-(k+1)}1 + (-1)^{k+1}P_{-k}0 = (-1)^k P_{-(k+1)}$$
$$h_{21} = (-1)^{k+1} P_{-k}2 + (-1)^{k}P_{-(k-1)}1 = (-1)^{k+1}\left(2P_{-k} - P_{-(k-1)}\right) = (-1)^{k+2}P_{-(k+1)}$$
$$h_{22} = (-1)^{k+1} P_{-k}1 + (-1)^{k}P_{-(k-1)}0 = (-1)^{k+1}P_{-k}$$

The identity $P_{-k} - 2P_{-(k+1)} = P_{-(k+2)}$ is used for the elements h_{11} and h_{21}.The element h_{11} of the matrix L^k is taken as $(-1)^k P_{-(k+1)}$. Therefore, if the $(-1)^2$ is added to the identities obtained using the element h_{11}, then they are found as $(-1)^{k+3}P_{-(k+2)}$ and $(-1)^{k+2}P_{-(k+1)}$.

Consequently, the matrix is given as follow:

$$L^{k+1} = \begin{pmatrix} (-1)^{k+3}P_{-(k+2)} & (-1)^{k+2}P_{-(k+1)} \\ (-1)^{k+2}P_{-(k+1)} & (-1)^{k+1}P_{-k} \end{pmatrix}$$

□

By using the matrices L^n and L^m, the following theorem is proven.

Theorem 5. For the negative indexed Pell numbers, the following identities are provided, where $n, m \geq 1$,

i. $P_{-(n+m+1)} = P_{-(n+1)}P_{-(m+1)} + P_{-n}P_{-m}$

ii. $P_{-(n+m)} = P_{-(n+1)}P_{-m} + P_{-n}P_{-(m-1)}$

iii. $P_{-(n+m)} = P_{-n}P_{-(m+1)} + P_{-(n-1)}P_{-m}$

iv. $P_{-(n+m-1)} = P_{-n}P_{-m} + P_{-(n-1)}P_{-(m-1)}$

v. $P_{-(2n+1)} = P^2_{-(n+1)} + P^2_{-n}$

vi. $P_{-2n} = P_{-n}\left[P_{-(n+1)} + P_{-(n-1)}\right]$

vii. $P_{-(2n-1)} = P^2_{-n} + P^2_{-(n-1)}$

viii. $2P_{-2n} = P^2_{-(n-1)} - P^2_{-(n+1)}$

ix. $P^2_{-(n-1)} + 2P^2_{-n} + P^2_{-(n+1)} = \frac{2P_{-4n}}{P_{-(2n-1)} - P_{-(2n+1)}}$

Proof. For $n, m \geq 1$, when the matrices L^n and L^m are written,

$$L^n = \begin{pmatrix} (-1)^{n+2}P_{-(n+1)} & (-1)^{n+1}P_{-n} \\ (-1)^{n+1}P_{-n} & (-1)^{n}P_{-(n-1)} \end{pmatrix}$$

and

$$L^m=\begin{pmatrix}(-1)^{m+2}P_{-(m+1)} & (-1)^{m+1}P_{-m}\\ (-1)^{m+1}P_{-m} & (-1)^{m}P_{-(m-1)}\end{pmatrix}$$

are obtained. From the equality of these matrices $L^{n+m}=L^nL^m$;

$$L^{n+m}=\begin{pmatrix}(-1)^{n+m+2}P_{-(n+m+1)} & (-1)^{n+m+1}P_{-(n+m)}\\ (-1)^{n+m+1}P_{-(n+m)} & (-1)^{n+m}P_{-(n+m-1)}\end{pmatrix}$$
$$=\begin{pmatrix}(-1)^{n+2}P_{-(n+1)} & (-1)^{n+1}P_{-n}\\ (-1)^{n+1}P_{-n} & (-1)^{n}P_{-(n-1)}\end{pmatrix}\begin{pmatrix}(-1)^{m+2}P_{-(m+1)} & (-1)^{m+1}P_{-m}\\ (-1)^{m+1}P_{-m} & (-1)^{m}P_{-(m-1)}\end{pmatrix}$$

is obtained.

i. From the equality of the elements at h_{11},

$$(-1)^{n+m+2}P_{-(n+m+1)}=(-1)^{n+2}P_{-(n+1)}(-1)^{m+2}P_{-(m+1)}+(-1)^{n+1}P_{-n}(-1)^{m+1}P_{-m}$$
$$=(-1)^{n+m+4}P_{-(n+1)}P_{-(m+1)}+(-1)^{n+m+2}P_{-n}P_{-m}=(-1)^{n+m}\left[P_{-(n+1)}P_{-(m+1)}+P_{-n}P_{-m}\right]$$

$$P_{-(n+m+1)}=P_{-(n+1)}P_{-(m+1)}+P_{-n}P_{-m}$$

is found.

ii. From the equality of the elements at h_{12},

$$(-1)^{n+m+1}P_{-(n+m)}=(-1)^{n+2}P_{-(n+1)}(-1)^{m+1}P_{-m}+(-1)^{n+1}P_{-n}(-1)^{m}P_{-(m-1)}$$
$$=(-1)^{n+m+3}P_{-(n+1)}P_{-m}+(-1)^{n+m+1}P_{-n}P_{-(m-1)}=(-1)^{n+m+1}\left[P_{-(n+1)}P_{-m}+P_{-n}P_{-(m-1)}\right]$$
$$P_{-(n+m)}=P_{-(n+1)}P_{-m}+P_{-n}P_{-(m-1)}$$

is obtained.

iii. From the equality of the elements at h_{21},

$$(-1)^{n+m+1}P_{-(n+m)}=(-1)^{n+1}P_{-n}(-1)^{m+2}P_{-(m+1)}+(-1)^{n}P_{-(n-1)}(-1)^{m+1}P_{-m}$$
$$=(-1)^{n+m+3}P_{-n}P_{-(m+1)}+(-1)^{n+m+1}P_{-(n-1)}P_{-m}=(-1)^{n+m+1}\left[P_{-n}P_{-(m+1)}+P_{-(n-1)}P_{-m}\right]$$
$$P_{-(n+m)}=P_{-n}P_{-(m+1)}+P_{-(n-1)}P_{-m}$$

holds.

iv. When the elements at h_{22} are compared,

$$(-1)^{n+m}P_{-(n+m-1)}=(-1)^{n+1}P_{-n}(-1)^{m+1}P_{-m}+(-1)^{n}P_{-(n-1)}(-1)^{m}P_{-(m-1)}$$
$$=(-1)^{n+m+2}P_{-n}P_{-m}+(-1)^{n+m}P_{-(n-1)}P_{-(m-1)}=(-1)^{n+m}\left[P_{-n}P_{-m}+P_{-(n-1)}P_{-(m-1)}\right]$$
$$P_{-(n+m-1)}=P_{-n}P_{-m}+P_{-(n-1)}P_{-(m-1)}$$

is obtained.

v. If $m=n$ is taken at i., then

$$P_{-(n+n+1)}=P_{-(n+1)}P_{-(n+1)}+P_{-n}P_{-n}$$
$$P_{-(2n+1)}=P_{-(n+1)}^2+P_{-n}^2$$

is provided.

vi. When $m=n$ is taken at ii., then

$$P_{-(n+n)}=P_{-(n+1)}P_{-n}+P_{-n}P_{-(n-1)}$$
$$P_{-2n}=P_{-n}\left[P_{-(n+1)}+P_{-(n-1)}\right]$$

is found.

vii. By taking $m=n$ at iv.,

$$P_{-(n+n-1)}=P_{-n}P_{-n}+P_{-(n-1)}P_{-(n-1)}$$
$$P_{-(2n-1)}=P_{-n}^2+P_{-(n-1)}^2$$

holds.

viii. From v and vii.,

$$P_{-(2n-1)}-P_{-(2n+1)}=\left[P_{-n}^2+P_{-(n-1)}^2\right]-\left[P_{-(n+1)}^2+P_{-n}^2\right]$$
$$=P_{-n}^2+P_{-(n-1)}^2-P_{-(n+1)}^2-P_{-n}^2$$
$$=P_{-(n-1)}^2-P_{-(n+1)}^2$$

is found. From the identity $P_{-(2n-1)}-P_{-(2n+1)}=2P_{-2n}$,

$$2P_{-2n}=P_{-(n-1)}^2-P_{-(n+1)}^2$$

is obtained.

ix. From v. and vii.,

$$P_{-(2n-1)}+P_{-(2n+1)}=\left[P_{-n}^2+P_{-(n-1)}^2\right]+\left[P_{-(n+1)}^2+P_{-n}^2\right]$$
$$=P_{-n}^2+P_{-(n-1)}^2+P_{-(n+1)}^2+P_{-n}^2$$
$$=P_{-(n-1)}^2+2P_{-n}^2+P_{-(n+1)}^2$$

is obtained. Also, the identity $2P_{-2n}=P_{-(n-1)}^2-P_{-(n+1)}^2$ is obtained at viii.,

$$2P_{-2n}=\left(P_{-(n-1)}+P_{-(n+1)}\right)\left(P_{-(n-1)}-P_{-(n+1)}\right)$$

Then,

$$\frac{2P_{-2n}}{P_{-(n-1)}-P_{-(n+1)}}=P_{-(n-1)}+P_{-(n+1)}$$

is found. In that case, for $n\to 2n$,

$$\frac{2P_{-4n}}{P_{-(2n-1)}-P_{-(2n+1)}}=P_{-(2n-1)}+P_{-(2n+1)}$$

is achieved. Consequently,

$$P^2_{-(n-1)}+2P^2_{-n}+P^2_{-(n+1)}=\frac{2P_{-4n}}{P_{-(2n-1)}-P_{-(2n+1)}}$$

holds.

Theorem 6. Let be $n\geq 1$ and P_n is the n^{th} Pell number.

$$R^n=\begin{pmatrix}\frac{(-1)^{3n-2}}{2}P_{-(2n-2)} & \frac{(-1)^{3n+1}}{2}P_{-2n}\\ \frac{(-1)^{3n+2}}{2}P_{-2n} & \frac{(-1)^{3n+3}}{2}P_{-(2n+2)}\end{pmatrix} \tag{7}$$

is given, where $R=\begin{pmatrix}0 & -1\\ 1 & -6\end{pmatrix}$.

Proof. By using the identity $P_{-n}=(-1)^{n+1}P_n$, $P_n=(-1)^{n+1}P_{-n}$is written. So, $P_{2n-2}=(-1)^{2n-1}P_{-(2n-2)}$, $P_{2n}=(-1)^{2n+1}P_{-2n}$ and $P_{2n+2}=(-1)^{2n+3}P_{-(2n+2)}$ holds. Let the elements of the matrix R^n are given by e_{ij} for$1\leq i,j\leq 2$. When, the elements of the matrix are written by negative indices, $e_{11}=\frac{(-1)^{n-1}}{2}(-1)^{2n-1}P_{-(2n-2)}=\frac{(-1)^{3n-2}}{2}P_{-(2n-2)}$,$e_{12}=\frac{(-1)^{n}}{2}(-1)^{2n+1}P_{-2n}=\frac{(-1)^{3n+1}}{2}P_{-2n}$, $e_{21}=\frac{(-1)^{n+1}}{2}(-1)^{2n+1}P_{-2n}=\frac{(-1)^{3n+2}}{2}P_{-2n}$ and $e_{22}=\frac{(-1)^{n}}{2}(-1)^{2n+3}P_{-(2n+2)}=\frac{(-1)^{3n+3}}{2}P_{-(2n+2)}$ are obtained. Consequently, the matrix R^n for the negative indexed Pell numbers is as follows;

$$\begin{pmatrix}\frac{(-1)^{3n-2}}{2}P_{-(2n-2)} & \frac{(-1)^{3n+1}}{2}P_{-2n}\\ \frac{(-1)^{3n+2}}{2}P_{-2n} & \frac{(-1)^{3n+3}}{2}P_{-(2n+2)}\end{pmatrix}.$$

□

Corollary 1. For $n\geq 1$, the determinant of the matrix R^n is given as

$$\det(R^n)=\frac{1}{4}\left[P^2_{-2n}-P_{-(2n-2)}P_{-(2n+2)}\right]. \tag{8}$$

Corollary 2. For $n\geq 1$, the trace of the matrix R^n is given as follows;

$$\text{trace } R^n=\frac{(-1)^{3n}}{2}\left[P_{-(2n-2)}-P_{-(2n+2)}\right]. \tag{9}$$

When the matrices R^n and R^m are examined according to the matrix theory, the following theorem presents new identities related to negative indexed Pell numbers.

Theorem 7. For the negative indexed Pell numbers the following identities are given, where $n,m\geq 1$;

i. $\frac{(-1)^{3(n+m)-2}}{2}P_{-(2(n+m)-2)}=\frac{(-1)^{3n+3m}}{4}\left[P_{-(2n-2)}P_{-(2m-2)}-P_{-2n}P_{-2m}\right]$
ii. $\frac{(-1)^{3(n+m)+1}}{2}P_{-2(n+m)}=\frac{(-1)^{3n+3m}}{4}\left[P_{-2n}P_{-(2m+2)}-P_{-(2n-2)}P_{-2m}\right]$
iii. $\frac{(-1)^{3(n+m)+2}}{2}P_{-2(n+m)}=\frac{(-1)^{3n+3m}}{4}\left[P_{-2n}P_{-(2m-2)}-P_{-(2n+2)}P_{-2m}\right]$
iv. $\frac{(-1)^{3(n+m)+3}}{2}P_{-(2(n+m)+2)}=\frac{(-1)^{3n+3m}}{4}\left[P_{-(2n+2)}P_{-(2m+2)}-P_{-2n}P_{-2m}\right]$
v. $P_{-(4n-2)}=\frac{1}{2}\left[P^2_{-(2n-2)}-P^2_{-2n}\right]$
vi. $P_{-4n}=-\frac{1}{2}\left[P_{-2n}P_{-(2n+2)}-P_{-(2n-2)}P_{-2n}\right]$
vii. $P_{-(4n+2)}=-\frac{1}{2}\left[P^2_{-(2n+2)}-P^2_{-2n}\right]$
viii. $P_{-(4n+1)}=-\frac{1}{2}\left[P_{-(2n+2)}P_{-(2n+1)}+P_{-2n}P_{-(2n-1)}\right]$

Proof. For $n,m\geq 1$, firstly, the matrices R^n and R^m are written as

$$R^n = \begin{pmatrix} \frac{(-1)^{3n-2}}{2}P_{-(2n-2)} & \frac{(-1)^{3n+1}}{2}P_{-2n} \\ \frac{(-1)^{3n+2}}{2}P_{-2n} & \frac{(-1)^{3n+3}}{2}P_{-(2n+2)} \end{pmatrix}$$

and

$$R^m = \begin{pmatrix} \frac{(-1)^{3m-2}}{2}P_{-(2m-2)} & \frac{(-1)^{3m+1}}{2}P_{-2m} \\ \frac{(-1)^{3m+2}}{2}P_{-2m} & \frac{(-1)^{3m+3}}{2}P_{-(2m+2)} \end{pmatrix}.$$

From the equation $R^{n+m} = R^n R^m$ in matrix theory,

$$R^{n+m} = \begin{pmatrix} \frac{(-1)^{3(n+m)-2}}{2}P_{-(2(n+m)-2)} & \frac{(-1)^{3(n+m)+1}}{2}P_{-2(n+m)} \\ \frac{(-1)^{3(n+m)+2}}{2}P_{-2(n+m)} & \frac{(-1)^{3(n+m)+3}}{2}P_{-(2(n+m)+2)} \end{pmatrix}$$
$$= \begin{pmatrix} \frac{(-1)^{3n-2}}{2}P_{-(2n-2)} & \frac{(-1)^{3n+1}}{2}P_{-2n} \\ \frac{(-1)^{3n+2}}{2}P_{-2n} & \frac{(-1)^{3n+3}}{2}P_{-(2n+2)} \end{pmatrix} \begin{pmatrix} \frac{(-1)^{3m-2}}{2}P_{-(2m-2)} & \frac{(-1)^{3m+1}}{2}P_{-2m} \\ \frac{(-1)^{3m+2}}{2}P_{-2m} & \frac{(-1)^{3m+3}}{2}P_{-(2m+2)} \end{pmatrix}$$

is attained.

i. When the elements are compared at e_{11},

$$\frac{(-1)^{3(n+m)-2}}{2}P_{-(2(n+m)-2)} = \frac{(-1)^{3n-2}}{2}P_{-(2n-2)}\frac{(-1)^{3m-2}}{2}P_{-(2m-2)} + \frac{(-1)^{3n+1}}{2}P_{-2n}\frac{(-1)^{3m+2}}{2}P_{-2m}$$
$$= \frac{(-1)^{3n+3m-4}}{4}P_{-(2n-2)}P_{-(2m-2)} + \frac{(-1)^{3n+3m+3}}{4}P_{-2n}P_{-2m}$$
$$= \frac{(-1)^{3n+3m}}{4}\left[P_{-(2n-2)}P_{-(2m-2)} - P_{-2n}P_{-2m}\right]$$

is obtained.

ii. From the equality of the elements of e_{12},

$$\frac{(-1)^{3(n+m)+1}}{2}P_{-2(n+m)} = \frac{(-1)^{3n-2}}{2}P_{-(2n-2)}\frac{(-1)^{3m+1}}{2}P_{-2m} + \frac{(-1)^{3n+1}}{2}P_{-2n}\frac{(-1)^{3m+3}}{2}P_{-(2m+2)}$$
$$= \frac{(-1)^{3n+3m-1}}{4}P_{-(2n-2)}P_{-2m} + \frac{(-1)^{3n+3m+4}}{4}P_{-2n}P_{-(2m+2)}$$
$$= \frac{(-1)^{3n+3m}}{4}\left[P_{-2n}P_{-(2m+2)} - P_{-(2n-2)}P_{-2m}\right]$$

results.

iii. From the comparison of the elements at e_{21},

$$\frac{(-1)^{3(n+m)+2}}{2}P_{-2(n+m)} = \frac{(-1)^{3n+2}}{2}P_{-2n}\frac{(-1)^{3m-2}}{2}P_{-(2m-2)} + \frac{(-1)^{3n+3}}{2}P_{-(2n+2)}\frac{(-1)^{3m+2}}{2}P_{-2m}$$
$$= \frac{(-1)^{3n+3m}}{4}P_{-2n}P_{-(2m-2)} + \frac{(-1)^{3n+3m+5}}{4}P_{-(2n+2)}P_{-2m}$$
$$= \frac{(-1)^{3n+3m}}{4}\left[P_{-2n}P_{-(2m-2)} - P_{-(2n+2)}P_{-2m}\right]$$

is derived.

iv. When the elements are compared at e_{22},

$$\frac{(-1)^{3(n+m)+3}}{2}P_{-(2(n+m)+2)} = \frac{(-1)^{3n+2}}{2}P_{-2n}\frac{(-1)^{3m+1}}{2}P_{-2m} + \frac{(-1)^{3n+3}}{2}P_{-(2n+2)}\frac{(-1)^{3m+3}}{2}P_{-(2m+2)}$$
$$= \frac{(-1)^{3n+3m+3}}{4}P_{-2n}P_{-2m} + \frac{(-1)^{3n+3m+6}}{4}P_{-(2n+2)}P_{-(2m+2)}$$
$$= \frac{(-1)^{3n+3m}}{4}\left[P_{-(2n+2)}P_{-(2m+2)} - P_{-2n}P_{-2m}\right]$$

is found.

v. If the elements at i. are taken as $m = n$, then

$$\frac{(-1)^{3(n+n)-2}}{2}P_{-(2(n+n)-2)} = \frac{(-1)^{3n+3n}}{4}\left[P_{-(2n-2)}P_{-(2n-2)} - P_{-2n}P_{-2n}\right]$$
$$P_{-(4n-2)} = \frac{1}{2}\left[P^2_{-(2n-2)} - P^2_{-2n}\right]$$

is attained.

vi. If $m = n$ is taken at ii.,

$$\frac{(-1)^{3(n+n)+1}}{2}P_{-2(n+n)}=\frac{(-1)^{3n+3n}}{4}\left[P_{-2n}P_{-(2n+2)}-P_{-(2n-2)}P_{-2n}\right]$$
$$P_{-4n}=-\frac{1}{2}\left[P_{-2n}P_{-(2n+2)}-P_{-(2n-2)}P_{-2n}\right]$$

is obtained.

vii. If $m=n$ is taken at iv.,

$$\frac{(-1)^{3(n+n)+3}}{2}P_{-(2(n+n)+2)}=\frac{(-1)^{3n+3n}}{4}\left[P_{-(2n+2)}P_{-(2n+2)}-P_{-2n}P_{-2n}\right]$$
$$P_{-(4n+2)}=-\frac{1}{2}\left[P_{-(2n+2)}^2-P_{-2n}^2\right]$$

is found.

viii. From vi. and vii.,

$$\begin{aligned}P_{-4n}-P_{-(4n+2)}&=-\frac{1}{2}\left[P_{-2n}P_{-(2n+2)}-P_{-(2n-2)}P_{-2n}\right]+\frac{1}{2}\left[P_{-(2n+2)}^2-P_{-2n}^2\right]\\&=\frac{1}{2}\left[P_{-(2n+2)}^2-P_{-2n}^2-P_{-2n}P_{-(2n+2)}+P_{-(2n-2)}P_{-2n}\right]\\&=\frac{1}{2}\left[\left(P_{-(2n+2)}^2-P_{-2n}P_{-(2n+2)}\right)+\left(P_{-(2n-2)}P_{-2n}-P_{-2n}^2\right)\right]\\&=\frac{1}{2}\left[P_{-(2n+2)}\left(P_{-(2n+2)}-P_{-2n}\right)+P_{-2n}\left(P_{-(2n-2)}-P_{-2n}\right)\right]\end{aligned}$$

is obtained. In addition, by using

$P_{-4n}-P_{-(4n+2)}=2P_{-(4n+1)}$, $P_{-(2n+2)}-P_{-2n}=-2P_{-(2n+1)}$ and $P_{-(2n-2)}-P_{-2n}=2P_{-(2n-1)}$ from the Pell identities,

$$2P_{-(4n+1)}=\frac{1}{2}\left[P_{-(2n+2)}\left(-2P_{-(2n+1)}\right)+P_{-2n}\left(2P_{-(2n-1)}\right)\right]$$
$$P_{-(4n+1)}=-\frac{1}{2}\left[P_{-(2n+2)}P_{-(2n+1)}+P_{-2n}P_{-(2n-1)}\right]$$

holds.

□

Theorem 8. If Q_n is the n^{th} Pell-Lucas number, for $n\geq 1$,

$$W_{-n}=\begin{pmatrix}-6 & 2\\ -2 & 6\end{pmatrix}\begin{pmatrix}0 & -1\\ 1 & -6\end{pmatrix}^n=\begin{pmatrix}2(-1)^{2n-1}Q_{-(2n-2)} & 2(-1)^{2n}Q_{-2n}\\ 2(-1)^{2n+1}Q_{-2n} & 2(-1)^{2n}Q_{-(2n+2)}\end{pmatrix}\qquad(10)$$

is given.

Proof. For $1\leq i,j\leq 2$, let the elements of matrix W_{-n} be l_{ij}. From the identity $l_{11}=2(-1)^{n-1}Q_{2n-2}=2(-1)^{2n-1}Q_{-(2n-2)}$, $l_{12}=2(-1)^nQ_{2n}=2(-1)^{2n}Q_{-2n}$,$l_{21}=2(-1)^{n+1}Q_{2n}=2(-1)^{2n+1}Q_{-2n}$ and $l_{22}=2(-1)^nQ_{2n+2}=2(-1)^{2n}Q_{-(2n+2)}$ is written and

$$W_{-n}=\begin{pmatrix}2(-1)^{2n-1}Q_{-(2n-2)} & 2(-1)^{2n}Q_{-2n}\\ 2(-1)^{2n+1}Q_{-2n} & 2(-1)^{2n}Q_{-(2n+2)}\end{pmatrix}$$

is found.

□

The following theorem gives new identities related to negative indexed Pell-Lucas numbers, by using the matrices W_{-n} and W_{-m}.

Theorem 9. For $n,m\geq 1$, the following identities are given for Pell-Lucas numbers,

i. $(-1)^{2(n+m)-1}Q_{-(2(n+m)-2)}=2(-1)^{2n+2m}\left[Q_{-(2n-2)}Q_{-(2m-2)}-Q_{-2n}Q_{-2m}\right]$
ii. $(-1)^{2(n+m)}Q_{-2(n+m)}=2(-1)^{2n+2m}\left[Q_{-2n}Q_{-(2m+2)}-Q_{-(2n-2)}Q_{-2m}\right]$
iii. $(-1)^{2(n+m)+1}Q_{-2(n+m)}=2(-1)^{2n+2m}\left[Q_{-2n}Q_{-(2m-2)}-Q_{-(2n+2)}Q_{-2m}\right]$
iv. $(-1)^{2(n+m)}Q_{-(2(n+m)+2)}=2(-1)^{2n+2m}\left[Q_{-(2n+2)}Q_{-(2m+2)}-Q_{-2n}Q_{-2m}\right]$
v. $Q_{-(4n-2)}=2\left[Q_{-2n}^2-Q_{-(2n-2)}^2\right]$
vi. $Q_{-4n}=2\left[Q_{-2n}Q_{-(2n+2)}-Q_{-(2n-2)}Q_{-2n}\right]$
vii. $Q_{-(4n+2)}=2\left[Q_{-(2n+2)}^2-Q_{-2n}^2\right]$
viii. $\frac{Q_{-(4n-2)}+Q_{-(4n+2)}}{2\left[Q_{-(2n+2)}+Q_{-(2n-2)}\right]}=Q_{-(2n+2)}-Q_{-(2n-2)}$

Proof. When the matrices W_{-n} and W_{-m} are written for $n,m\geq 1$,

$$W_{-n}=\begin{pmatrix}2(-1)^{2n-1}Q_{-(2n-2)} & 2(-1)^{2n}Q_{-2n}\\ 2(-1)^{2n+1}Q_{-2n} & 2(-1)^{2n}Q_{-(2n+2)}\end{pmatrix}$$

and

$$W_{-m}=\begin{pmatrix}2(-1)^{2m-1}Q_{-(2m-2)} & 2(-1)^{2m}Q_{-2m}\\ 2(-1)^{2m+1}Q_{-2m} & 2(-1)^{2m}Q_{-(2m+2)}\end{pmatrix}$$

are found. From the equation in matrix theory $W_{-(n+m)}=W_{-n}W_{-m}$,

$$\begin{aligned}W_{-(n+m)}&=\begin{pmatrix}2(-1)^{2(n+m)-1}Q_{-(2(n+m)-2)} & 2(-1)^{2(n+m)}Q_{-2(n+m)}\\ 2(-1)^{2(n+m)+1}Q_{-2(n+m)} & 2(-1)^{2(n+m)}Q_{-(2(n+m)+2)}\end{pmatrix}\\ &=\begin{pmatrix}2(-1)^{2n-1}Q_{-(2n-2)} & 2(-1)^{2n}Q_{-2n}\\ 2(-1)^{2n+1}Q_{-2n} & 2(-1)^{2n}Q_{-(2n+2)}\end{pmatrix}\begin{pmatrix}2(-1)^{2m-1}Q_{-(2m-2)} & 2(-1)^{2m}Q_{-2m}\\ 2(-1)^{2m+1}Q_{-2m} & 2(-1)^{2m}Q_{-(2m+2)}\end{pmatrix}\end{aligned}$$

is obtained. When the elements of these matrices are compared, the following items are produced.

i. From the equality of the elements at l_{11},

$$\begin{aligned}2(-1)^{2(n+m)-1}Q_{-(2(n+m)-2)}&=2(-1)^{2n-1}Q_{-(2n-2)}2(-1)^{2m-1}Q_{-(2m-2)}+2(-1)^{2n}Q_{-2n}2(-1)^{2m+1}Q_{-2m}\\ &=4(-1)^{2n+2m-2}Q_{-(2n-2)}Q_{-(2m-2)}+4(-1)^{2n+2m+1}Q_{-2n}Q_{-2m}\\ &=4(-1)^{2n+2m}\left[Q_{-(2n-2)}Q_{-(2m-2)}-Q_{-2n}Q_{-2m}\right]\\ (-1)^{2(n+m)-1}Q_{-(2(n+m)-2)}&=2(-1)^{2n+2m}\left[Q_{-(2n-2)}Q_{-(2m-2)}-Q_{-2n}Q_{-2m}\right]\end{aligned}$$

is found.

ii. When the elements at l_{12} are equated, then

$$\begin{aligned}2(-1)^{2(n+m)}Q_{-2(n+m)}&=2(-1)^{2n-1}Q_{-(2n-2)}2(-1)^{2m}Q_{-2m}+2(-1)^{2n}Q_{-2n}2(-1)^{2m}Q_{-(2m+2)}\\ &=4(-1)^{2n+2m-1}Q_{-(2n-2)}Q_{-2m}+4(-1)^{2n+2m}Q_{-2n}Q_{-(2m+2)}\\ &=4(-1)^{2n+2m}\left[Q_{-2n}Q_{-(2m+2)}-Q_{-(2n-2)}Q_{-2m}\right]\\ (-1)^{2(n+m)}Q_{-2(n+m)}&=2(-1)^{2n+2m}\left[Q_{-2n}Q_{-(2m+2)}-Q_{-(2n-2)}Q_{-2m}\right]\end{aligned}$$

is obtained.

iii. If the elements at l_{21} are compared, then

$$\begin{aligned}&2(-1)^{2(n+m)+1}Q_{-2(n+m)}=2(-1)^{2n+1}Q_{-2n}2(-1)^{2m-1}Q_{-(2m-2)}+2(-1)^{2n}Q_{-(2n+2)}2(-1)^{2m+1}Q_{-2m}\\ &=4(-1)^{2n+2m}Q_{-2n}Q_{-(2m-2)}+4(-1)^{2n+2m+1}Q_{-(2n+2)}Q_{-2m}=4(-1)^{2n+2m}\left[Q_{-2n}Q_{-(2m-2)}-Q_{-(2n+2)}Q_{-2m}\right]\\ &(-1)^{2(n+m)+1}Q_{-2(n+m)}=2(-1)^{2n+2m}\left[Q_{-2n}Q_{-(2m-2)}-Q_{-(2n+2)}Q_{-2m}\right]\end{aligned}$$

holds.

iv. From the equality of the elements at l_{22},

$$\begin{aligned}2(-1)^{2(n+m)}Q_{-(2(n+m)+2)}&=2(-1)^{2n+1}Q_{-2n}2(-1)^{2m}Q_{-2m}+2(-1)^{2n}Q_{-(2n+2)}2(-1)^{2m}Q_{-(2m+2)}\\ &=4(-1)^{2n+2m+1}Q_{-2n}Q_{-2m}+4(-1)^{2n+2m}Q_{-(2n+2)}Q_{-(2m+2)}\\ &=4(-1)^{2n+2m}\left[Q_{-(2n+2)}Q_{-(2m+2)}-Q_{-2n}Q_{-2m}\right]\\ (-1)^{2(n+m)}Q_{-(2(n+m)+2)}&=2(-1)^{2n+2m}\left[Q_{-(2n+2)}Q_{-(2m+2)}-Q_{-2n}Q_{-2m}\right]\end{aligned}$$

is attained.

v. If $m=n$ is taken at i., then

$$\begin{aligned}(-1)^{2(n+n)-1}Q_{-(2(n+n)-2)}&=2(-1)^{2n+2n}\left[Q_{-(2n-2)}Q_{-(2n-2)}-Q_{-2n}Q_{-2n}\right]\\ (-1)^{4n-1}Q_{-(4n-2)}&=2(-1)^{4n}\left[Q^2_{-(2n-2)}-Q^2_{-2n}\right]\\ Q_{-(4n-2)}&=2\left[Q^2_{-2n}-Q^2_{-(2n-2)}\right]\end{aligned}$$

is obtained.

vi. When $m=n$ is taken at ii., then

$$\begin{aligned}(-1)^{2(n+n)}Q_{-2(n+n)}&=2(-1)^{2n+2n}\left[Q_{-2n}Q_{-(2n+2)}-Q_{-(2n-2)}Q_{-2n}\right]\\ Q_{-4n}&=2\left[Q_{-2n}Q_{-(2n+2)}-Q_{-(2n-2)}Q_{-2n}\right]\end{aligned}$$

holds.

vii. At iv., $m=n$ is taken,

$$\begin{aligned}(-1)^{2(n+n)}Q_{-(2(n+n)+2)}&=2(-1)^{2n+2n}\left[Q_{-(2n+2)}Q_{-(2n+2)}-Q_{-2n}Q_{-2n}\right]\\ Q_{-(4n+2)}&=2\left[Q^2_{-(2n+2)}-Q^2_{-2n}\right]\end{aligned}$$

is obtained.

viii. From v. and vii.,

$$\begin{aligned}Q_{-(4n-2)}+Q_{-(4n+2)}&=2\left[Q^2_{-2n}-Q^2_{-(2n-2)}\right]+2\left[Q^2_{-(2n+2)}-Q^2_{-2n}\right]\\ &=2Q^2_{-2n}-2Q^2_{-(2n-2)}+2Q^2_{-(2n+2)}-2Q^2_{-2n}\\ &=2Q^2_{-(2n+2)}-2Q^2_{-(2n-2)}\\ &=2\left(Q_{-(2n+2)}+Q_{-(2n-2)}\right)\left(Q_{-(2n+2)}-Q_{-(2n-2)}\right)\end{aligned}$$

$$\frac{Q_{-(4n-2)}+Q_{-(4n+2)}}{2\left[Q_{-(2n+2)}+Q_{-(2n-2)}\right]}=Q_{-(2n+2)}-Q_{-(2n-2)}$$

is produced.

Conclusion

In this paper, the Pell and Pell-Lucas number sequences are examined in detail. Using the Binet formulas associated with these sequences, matrices with negative indices and their corresponding power representations are derived, along with related identities. The results presented in this paper are significant due to their broad applicability in many areas such as optimization problems and prediction algorithms.

Conflicts of interest

There are no conflicts of interest in this work.

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