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SOME FIXED POINT RESULTS FOR A NEW THREE STEPS ITERATION PROCESS IN BANACH SPACES

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Abstract. In this paper, we introduce a three step iteration method and show that this method can be used to approximate fixed point of weak contraction mappings. Furthermore, we prove that this iteration method is equivalent to Mann iterative scheme and converges faster than Picard-S iterative scheme for the class of weak contraction mappings. We also present tables and three graphics to support this result. Finally, we prove a data dependence result for weak contraction mappings using this three step iterative scheme.
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1. Introduction

Let X be a Banach space, and C be a nonempty, closed, convex subset of X. Let T be a mapping from a set C to itself. An element x in C is said to be a fixed point of T if $Tx = x$.

The iterative approximation of a fixed point for certain classes of operators is one of the main tools in the fixed point theory. Therefore, a lot of iterative methods have been defined and studied by numerous mathematicians (see [2], [5], [6], [7], [9]-[13], [15], [16]).

Recently, Gürsoy and Karakaya [8] introduced Picard-S iterative process as follows:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = Ty_n \\ y_n = (1-\alpha_n)\, Tx_n + \alpha_n Tz_n \\ z_n = (1-\beta_n)\, x_n + \beta_n Tx_n \ (n \in \mathbb{N}), \end{cases} \tag{1.1}$$

where $(\alpha_n)_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty} \in [0,1]$.

Lemma 1.1. [19] *Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be nonnegative real sequences satisfying the following condition:*

$$a_{n+1} \leq (1-\mu_n)a_n + b_n, \tag{1.2}$$

where $\mu_n \in (0,1)$ for all $n \geq n_0$, $\sum\limits_{n=1}^{\infty} \mu_n = \infty$ and $\frac{b_n}{\mu_n} \to 0$ as $n \to \infty$. Then $\lim_{n\to\infty} a_n = 0$.

Lemma 1.2. [18] *Let $\{a_n\}_{n=1}^{\infty}$ be a nonnegative real sequence and there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ satisfying the following condition:*

$$a_{n+1} \leq (1-\mu_n)a_n + \mu_n \eta_n, \tag{1.3}$$

where $\mu_n \in (0,1)$ such that $\sum\limits_{n=1}^{\infty} \mu_n = \infty$ and $\eta_n \geq 0$. Then the following inequality holds:

$$0 \leq \lim_{n\to\infty} \sup a_n \leq \lim_{n\to\infty} \sup \eta_n. \tag{1.4}$$

Definition 1.3. [4] *The self-map $T : C \to C$ is called weak-contraction if there exist $\delta \in (0,1)$ and $L \geq 0$ such that*

$$\|Tx - Ty\| \leq \delta \|x - y\| + L \|y - Tx\|.$$

Theorem 1.4. [4] *Let (X, d) be a complete metric space and $T : X \to X$ be a weak contraction for which there exist $\delta \in (0, 1)$ and some $L_1 \geq 0$ such that*

$$\|Tx - Ty\| \leq \delta \|x - y\| + L_1 \|x - Tx\|. \tag{1.5}$$

Then, T has a unique fixed point.

Definition 1.5. [3] *Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be nonnegative real convergent sequences with limits a and b respectively. Then, $\{a_n\}_{n=1}^{\infty}$ converges faster than $\{b_n\}_{n=1}^{\infty}$ if*

$$\lim_{n\to\infty} \left| \frac{a_n - a}{b_n - b} \right| = 0. \tag{1.6}$$

Definition 1.6. [18] *Let T, $S : C \to C$ be two operators. We say that S is an approximate operator of T for all $x \in C$ and a fixed $\varepsilon > 0$ if $\|Tx - Sx\| \leq \varepsilon$.*

In this paper, we introduce the following new iterative scheme:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = Ty_n \\ y_n = (1-\alpha_n)\, z_n + \alpha_n T z_n \\ z_n = T x_n \ (n \in \mathbb{N}) \end{cases} \tag{1.7}$$

where $(\alpha_n)_{n=1}^{\infty} \in [0,1]$ and $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$.

2. Main results

Theorem 2.1. *Let C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ be a weak-contraction map satisfying condition (1.5). Let $\{x_n\}_{n=1}^{\infty}$ be an iterative sequence generated by (1.7) with a real sequence $\{\alpha_n\}_{n=1}^{\infty} \in [0,1]$ satisfying $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$. Then, $\{x_n\}_{n=1}^{\infty}$ converges to a unique fixed point p_* of T.*

Proof. We will follow the same scheme of proof as for a similar result on overlaps ([8], Theorem 1). It can easily be seen from (1.5) that p_* is the unique fixed point of T. We have to show that $x_n \to p_*$ as $n \to \infty$. From (1.7) and (1.5), we have

$$\|z_n - p_*\| = \|Tx_n - p_*\| \le \delta \|x_n - p_*\|,$$

and

$$\begin{aligned} \|y_n - p_*\| &= \|(1-\alpha_n)\, z_n + \alpha_n T z_n - p_*\| \\ &\le (1-\alpha_n)\|z_n - p_*\| + \alpha_n \|Tz_n - Tp_*\| \\ &\le \delta[1-\alpha_n(1-\delta)]\, \|x_n - p_*\|. \end{aligned}$$

Then, we obtain

$$\begin{aligned} \|x_{n+1} - p_*\| &= \|Ty_n - p_*\| \le \delta \|y_n - p_*\| \\ &\le \delta^2[1-\alpha_n(1-\delta)]\, \|x_n - p_*\|. \end{aligned}$$

Repeating this process n-time, we obtain the following inequalities:

$$\begin{aligned} \|x_n - p_*\| &\le \delta^2[1-\alpha_{n-1}(1-\delta)]\, \|x_{n-1} - p_*\| \\ \|x_{n-1} - p_*\| &\le \delta^2[1-\alpha_{n-2}(1-\delta)]\, \|x_{n-2} - p_*\| \\ &\ \ ... \\ \|x_1 - p_*\| &\le \delta^2[1-\alpha_1(1-\delta)]\, \|x_1 - p_*\|. \end{aligned} \tag{2.1}$$

From inequalities (2.1), we have

$$\|x_{n+1} - p_*\| \le \|x_1 - p_*\|\, \delta^{2n} \prod_{i=1}^{n} [1-\alpha_i(1-\delta)]. \tag{2.2}$$

Since $\delta \in (0,1)$, we obtain $[1-\alpha_n(1-\delta)] < 1$.

From classical analysis, we know that $1-x \leq e^{-x}$ for all $x \in [0,1]$. By using this inequality with (2.2), we obtain

$$\begin{aligned} \|x_{n+1}-p_*\| &\leq \|x_1-p_*\| \, \delta^{2n} \prod_{i=1}^{n} e^{-(1-\delta)\alpha_i} \\ &= \|x_1-p_*\| \, \delta^{2n} e^{-(1-\delta)\sum\limits_{i=1}^{n}\alpha_i}. \end{aligned} \tag{2.3}$$

Taking the limit in both sides of inequality (2.3), it can be seen that $x_n \to p_*$ as $n \to \infty$.

Theorem 2.2. *Let X be a Banach space, C be a nonempty, closed, convex subset of X and $T : C \to C$ be a weak-contraction map satisfying condition (1.5) with a fixed point p_*. Let $\{u_n\}_{n=1}^{\infty}$ be the Mann iteration process defined in* [12] *with $u_1 \in C$ and $\{x_n\}_{n=1}^{\infty}$ defined by (1.7) with $x_1 \in C$ and a real sequence $\{\alpha_n\}_{n=1}^{\infty} \in$ [0,1] satisfying $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$. Then the following assertions are equivalent:*

i) The Mann (see [12]*) iteration converges to p_*.*
ii) The new iteration method (1.7) converges to p_*.

Proof. We will show that $(i) \Rightarrow (ii)$, that is, if the Mann iteration method converges, then the iteration method (1.7) does too. Now, by using Mann iteration and (1.7) we have

$$\begin{aligned} \|u_{n+1}-x_{n+1}\| &= \|(1-\alpha_n)\, u_n + \alpha_n T u_n - T y_n\| \\ &\leq (1-\alpha_n)\|u_n - Ty_n\| + \alpha_n \|Tu_n - Ty_n\| \\ &\leq (1-\alpha_n)\{\|u_n - Tu_n\| + \|Tu_n - Ty_n\|\} + \alpha_n \|Tu_n - Ty_n\| \\ &\leq [1-\alpha_n + L]\|u_n - Tu_n\| + \delta \|u_n - y_n\|, \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} \|u_n - y_n\| &\leq (1-\alpha_n)\|u_n - z_n\| + \alpha_n \|u_n - Tz_n\| \\ &\leq (1-\alpha_n)\|u_n - z_n\| + \alpha_n\{\|u_n - Tu_n\| \\ &\quad + \|Tu_n - Tz_n\|\} \\ &\leq [1-\alpha_n(1-\delta)]\|u_n - z_n\| + \alpha_n(1+L)\|u_n - Tu_n\|, \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} \|u_n - z_n\| &\leq \|u_n - Tu_n\| + \|Tu_n - Tx_n\| \\ &\leq \delta \|u_n - x_n\| + (1+L)\|u_n - Tu_n\|. \end{aligned} \tag{2.6}$$

Substituting (2.6) in (2.5) and (2.5) in (2.4) respectively, we obtain

$$\begin{aligned} \|u_{n+1}-x_{n+1}\| &\leq \{1-\alpha_n + L + \alpha_n\delta(1+L) \\ &\quad +\delta[1-\alpha_n(1-\delta)](1+L)\}\|u_n - Tu_n\| \\ &\quad +\delta^2[1-\alpha_n(1-\delta)]\|u_n - x_n\| \\ &\leq \{1-\alpha_n + L + \delta(1+L)(1+\alpha_n\delta)\}\|u_n - Tu_n\| \\ &\quad +[1-\alpha_n(1-\delta)]\|u_n - x_n\|. \end{aligned}$$

Let

$$\begin{aligned}
\mu_n &= \alpha_n(1-\delta) \in (0,1) \\
a_n &= \|u_n - x_n\| \\
b_n &= \{1-\alpha_n + L + \delta\,(1+L)\,(1+\alpha_n\delta)\}\,\|u_n - Tu_n\|\,.
\end{aligned}$$

Furthermore using $Tp_* = p_*$ and $\|u_n - p_*\| \to 0$, we have

$$\begin{aligned}
\|u_n - Tu_n\| &= \|u_n - p_* + Tp_* - Tu_n\| \\
&\le \|u_n - p_*\| + \delta\,\|u_n - p_*\| + L\,\|p_* - Tp_*\| \\
&= (1+\delta)\,\|u_n - p_*\|\,.
\end{aligned}$$

Then, $\|u_n - Tu_n\| \to 0$. Because of these results, we obtain $b_n \to 0$. By applying Lemma 1.1, we have $a_n = \|u_n - x_n\| \to 0$ as $n \to \infty$. Consequently,

$$\|u_{n+1} - x_{n+1}\| \to 0 \text{ as } n \to \infty.$$

Now, we show that $(ii) \to (i)$:

$$\begin{aligned}
\|x_{n+1} - u_{n+1}\| &\le (1-\alpha_n)\,\|Ty_n - u_n\| + \alpha_n\,\|Ty_n - Tu_n\| \\
&\le (1-\alpha_n + \alpha_n L)\,\|y_n - Ty_n\| + [1-\alpha_n(1-\delta)]\,\|y_n - u_n\|\,,
\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}
\|y_n - u_n\| &= (1-\alpha_n)\,\|z_n - u_n\| + \alpha_n\,\|Tz_n - u_n\| \\
&\le (1-\alpha_n)\,\|z_n - u_n\| + \alpha_n\,\|Tz_n - z_n\| + \alpha_n\,\|z_n - u_n\| \\
&= \|z_n - u_n\| + \alpha_n\,\|Tz_n - z_n\|\,,
\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}
\|z_n - u_n\| &= \|Tx_n - u_n\| \\
&\le \|Tx_n - x_n\| + \|x_n - u_n\|\,.
\end{aligned} \tag{2.9}$$

Substituting (2.9) in (2.8) and (2.8) in (2.7) respectively, we obtain

$$\begin{aligned}
\|x_{n+1} - u_{n+1}\| \le\ & [1-\alpha_n(1-\delta)]\,\|x_n - u_n\| \\
& +[1-\alpha_n(1-\delta)]\,\|x_n - Tx_n\| \\
& +(1-\alpha_n + \alpha_n L)\,\|y_n - Ty_n\| \\
& +[1-\alpha_n(1-\delta)]\alpha_n\,\|z_n - Tz_n\|\,.
\end{aligned} \tag{2.10}$$

Using $Tp_* = p_*$ and $\|x_n - p_*\| \to 0$ as $n \to \infty$, we have

$$\begin{aligned}
\|x_n - Tx_n\| &\le \|x_n - p_*\| + \|Tp_* - Tx_n\| \\
&\le \|x_n - p_*\| + \delta\,\|x_n - p_*\| + L\,\|p_* - Tp_*\| \\
&= (1+\delta)\,\|x_n - p_*\|\,.
\end{aligned}$$

Then, $\|x_n - Tx_n\| \to 0$ as $n \to \infty$. Similarly,

$$\begin{aligned}
\|y_n - Ty_n\| &\leq \|y_n - p_*\| + \|Tp_* - Ty_n\| \\
&\leq \|y_n - p_*\| + \delta \|y_n - p_*\| + L \|p_* - Tp_*\| \\
&= (1+\delta) \|y_n - p_*\| \\
&= (1+\delta) \|(1-\alpha_n) z_n + \alpha_n T z_n - p_*\| \\
&\leq (1+\delta)(1-\alpha_n) \|z_n - p_*\| + (1+\delta)\alpha_n \|Tz_n - Tp_*\| \\
&\leq (1+\delta)[1-\alpha_n(1-\delta)] \|z_n - p_*\|,
\end{aligned}$$

and

$$\|z_n - p_*\| = \|Tx_n - p_*\| \leq \delta \|x_n - p_*\|,$$

then $\|z_n - p_*\| \to 0$ as $n \to \infty$. Thus, $\|y_n - Ty_n\| \to 0$ as $n \to \infty$. Moreover,

$$\begin{aligned}
\|z_n - Tz_n\| &\leq \|z_n - p_*\| + \|Tp_* - Tz_n\| \\
&\leq \|z_n - p_*\| + \delta \|z_n - p_*\| + L \|p_* - Tp_*\| \\
&= (1+\delta) \|z_n - p_*\|,
\end{aligned}$$

and hence $\|z_n - Tz_n\| \to 0$ as $n \to \infty$. Denote,

$$\begin{aligned}
\mu_n &= \alpha_n(1-\delta) \in (0,1) \\
a_n &= \|x_n - u_n\| \\
b_n &= [1-\alpha_n(1-\delta)] \|x_n - Tx_n\| \\
&\quad + (1-\alpha_n + \alpha_n L) \|y_n - Ty_n\| \\
&\quad + [1-\alpha_n(1-\delta)]\alpha_n \|z_n - Tz_n\|.
\end{aligned}$$

Thus, from Lemma 1.1, $a_n = \|x_n - u_n\| \to 0$ as $n \to \infty$. From (2.10),

$$\|x_{n+1} - u_{n+1}\| \to 0 \text{ as } n \to \infty.$$

Corollary 2.3. *Let X be a Banach space, C be a nonempty, closed, convex subset of X and $T : C \to C$ be a weak contraction mapping satisfying condition (1.5) with a fixed point p_*. If the initial point is the same for all iterations, then the following assertions are equivalent:*

(1) *the Picard-S iterative scheme (1.1) converges to p_*,*

(2) *the new iteration (1.7) converges to p_*,*

(3) *the CR iteration (see* [5]*) converges to p_*,*

(4) *the Ishikawa iteration (see* [9]*) converges to p_*,*

(5) *the S^* iteration (see* [10]*) converges to p_*,*

(6) *the Mann iteration (see* [12]*) converges to p_*,*

(7) *the Noor iteration (see* [13]*) converges to p_*,*

(8) *the SP iteration (see* [15]*) converges to p_*,*

(9) *the Picard iteration (see* [16]*) converges to p_*.*

In the following theorem, we compare the rate of convergence of iterative scheme (1.7) and Picard-S iterative process (1.1). Also, in order to support the analytical proof of Theorem 2.4 and to demonstrate the efficiency of new iteration (1.7), we give some numerical examples.

Theorem 2.4. *Let X be a Banach space, and C be a closed, convex subset of X, and $T : C \to C$ be a weak contraction mapping satisfying condition (1.5) with a fixed point p_*. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in [0,1] satisfying (*) $\alpha_1 \leq \alpha_n \leq 1$, $\beta_1 \leq \beta_n \leq 1$ for all $n \in \mathbb{N}$ and some α_1, $\alpha_2 > 0$. For given $u_1 = x_1 \in C$, consider the iterative sequences $\{x_n\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ defined by (1.7) and (1.1) respectively. Then, $\{x_n\}_{n=1}^{\infty}$ converges to p_* faster than $\{u_n\}_{n=1}^{\infty}$ does.*

Proof. From (2.2) in Theorem 2.1, we have the following inequality

$$\|x_{n+1} - p_*\| \leq \|x_1 - p_*\| \, \delta^{2n} \prod_{i=1}^{n} [1 - \alpha_i(1-\delta)]. \tag{2.11}$$

It is easy to see that,

$$\|u_{n+1} - p_*\| \leq \|u_1 - p_*\| \, \delta^{2n} \prod_{i=1}^{n} [1 - \alpha_i\beta_i(1-\delta)]. \tag{2.12}$$

Applying assumption (*) to (2.11) and (2.12) respectively, we obtain

$$\begin{aligned} \|x_{n+1} - p_*\| &\leq \|x_1 - p_*\| \, \delta^{2n} \prod_{i=1}^{n} [1 - \alpha_1(1-\delta)] \qquad (2.13) \\ &= \|x_1 - p_*\| \, \delta^{2n} [1 - \alpha_1(1-\delta)]^n, \end{aligned}$$

$$\begin{aligned} \|u_{n+1} - p_*\| &\leq \|u_1 - p_*\| \, \delta^{2n} \prod_{i=1}^{n} [1 - \alpha_1\beta_1(1-\delta)] \qquad (2.14) \\ &= \|u_1 - p_*\| \, \delta^{2n} [1 - \alpha_1\beta_1(1-\delta)]^n. \end{aligned}$$

Define

$$\begin{aligned} a_n &= \|x_1 - p_*\| \, \delta^{2n} [1 - \alpha_1(1-\delta)]^n, \\ b_n &= \|u_1 - p_*\| \, \delta^{2n} [1 - \alpha_1\beta_1(1-\delta)]^n, \end{aligned}$$

and

$$\begin{aligned} \psi_n &= \frac{a_n}{b_n} = \frac{\|x_1 - p_*\| \, \delta^{2n} [1 - \alpha_1(1-\delta)]^n}{\|u_1 - p_*\| \, \delta^{2n} [1 - \alpha_1\beta_1(1-\delta)]^n} \\ &= \left(\frac{1 - \alpha_1(1-\delta)}{1 - \alpha_1\beta_1(1-\delta)} \right)^n. \end{aligned}$$

Since δ and $\beta_1 \in (0,1)$, we have

$$\begin{aligned} \beta_1 &< 1 \\ &\Rightarrow \alpha_1\beta_1 < \alpha_1 \\ &\Rightarrow \alpha_1\beta_1(1-\delta) < \alpha_1(1-\delta) \\ &\Rightarrow \frac{[1 - \alpha_1(1-\delta)]}{[1 - \alpha_1\beta_1(1-\delta)]} < 1. \end{aligned}$$

Therefore, $\lim_{n\to\infty}\psi_n = 0$. From Definition 1.5, we obtain that $\{x_n\}_{n=1}^{\infty}$converges faster than $\{u_n\}_{n=1}^{\infty}$.

Example 2.5. Let $X = \mathbb{R}$ and $C = [1, \infty)$. Let $T : C \to C$ be a mapping defined by $T(x) = \frac{3}{4}(x + \frac{1}{x})$ for all $x \in C$. It is easy to show that T is a weak contraction with fixed point $p_* = 1,73205080756888$. Choose $\alpha_n = \beta_n = \gamma_n = \frac{1}{4}$ with the initial value $x_1 = 1$. The following Tables 1-2-3 show that the new iteration method (1.7) converges faster than all S [2], CR [5], Ishikawa [9], S* [10], Mann [12], Noor [13], SP [15], Picard [16], Picard-Mann [17], Picard-S (1.1) iteration methods including the iteration method due to Abbas and Nazir [1].

Table 1. Comparison rate of convergence among some iteration methods

x_n	Mann	Ishikawa	Noor	SP
x_1	1	1	1	1
x_2	1,12500000000000	1,12760416666667	1,12770753644630	1,29853765491738
x_3	1,22135416666667	1,22835269106877	1,22874546233073	1,45852260608018
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{83}	1,73204106834632	1,73204843765511	1,73204881698286	1,73205080756887
x_{84}	1,73204228575256	1,73204877092441	1,73204910079701	1,73205080756888
x_{85}	1,73204335098222	1,73204905732768	1,73204934414543	1,73205080756888
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{218}	1,73205080756873	1,73205080756887	1,73205080756888	1,73205080756888
x_{219}	1,73205080756875	1,73205080756887	1,73205080756888	1,73205080756888
x_{220}	1,73205080756877	1,73205080756887	1,73205080756888	1,73205080756888
x_{221}	1,73205080756877	1,73205080756887	1,73205080756888	1,73205080756888
x_{222}	1,73205080756879	1,73205080756888	1,73205080756888	1,73205080756888
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{249}	1,73205080756887	1,73205080756888	1,73205080756888	1,73205080756888
x_{250}	1,73205080756888	1,73205080756888	1,73205080756888	1,73205080756888

Table 1 shows that SP iteration reaches the fixed point at the 84^{th} step while Noor, Ishikawa and Mann iterations reach 218^{th}step, 221^{th} step, and 250^{th} step, respectively.

Table 2. Comparison rate of convergence among some iteration methods

x_n	Picard	Picard-Mann	S^*	S
x_1	1	1	1	1
x_2	1,50000000000000	1,51041666666667	1,53152164346837	1,50260416666667
x_3	1,62500000000000	1,64207959798239	1,65074040007186	1,62936277984372
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{38}	1,73205080756595	1,73205080756885	1,73205080756886	1,73205080756795
x_{39}	1,73205080756742	1,73205080756887	1,73205080756887	1,73205080756843
x_{40}	1,73205080756815	1,73205080756887	1,73205080756887	1,73205080756866
x_{41}	1,73205080756851	1,73205080756888	1,73205080756888	1,73205080756877
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{46}	1,73205080756887	1,73205080756888	1,73205080756888	1,73205080756887
x_{47}	1,73205080756887	1,73205080756888	1,73205080756888	1,73205080756888
x_{48}	1,73205080756887	1,73205080756888	1,73205080756888	1,73205080756888
x_{49}	1,73205080756888	1,73205080756888	1,73205080756888	1,73205080756888

Table 2 shows that Picard iteration reaches the fixed point at the 49^{th} step while S iteration reaches at the 47^{th} step, Picard-Mann and S^* iterations reach at the 41^{th}step.

Table 3. Comparison rate of convergence among some iteration methods

x_n	Our iteration	Picard-S	Abbas and Nazır	CR
x_1	1	1	1	1
x_2	1,63823341836735	1,62608657387348	1,59716909707178	1,53347476846837
x_3	1,71238829126157	1,70759059791304	1,69489357952688	1,65363032158024
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{22}	1,73205080756887	1,73205080756883	1,73205080756650	1,73205080137048
x_{23}	1,73205080756888	1,73205080756887	1,73205080756819	1,73205080494182
x_{24}	1,73205080756888	1,73205080756887	1,73205080756868	1,73205080645546
x_{25}	1,73205080756888	1,73205080756888	1,73205080756882	1,73205080709698
x_{26}	1,73205080756888	1,73205080756888	1,73205080756886	1,73205080736887
x_{27}	1,73205080756888	1,73205080756888	1,73205080756887	1,73205080748411
x_{28}	1,73205080756888	1,73205080756888	1,73205080756888	1,73205080753295
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{39}	1,73205080756888	1,73205080756888	1,73205080756888	1,73205080756887
x_{40}	1,73205080756888	1,73205080756888	1,73205080756888	1,73205080756888

Table 3 shows that our iteration reaches fixed point at the 23^{th} step while Abbas and Nazir, Picard-S and CR iterations reach 28^{th}step, 25^{th}step, 40^{th}step, respectively.

The following figures are graphical presentations of the above results:

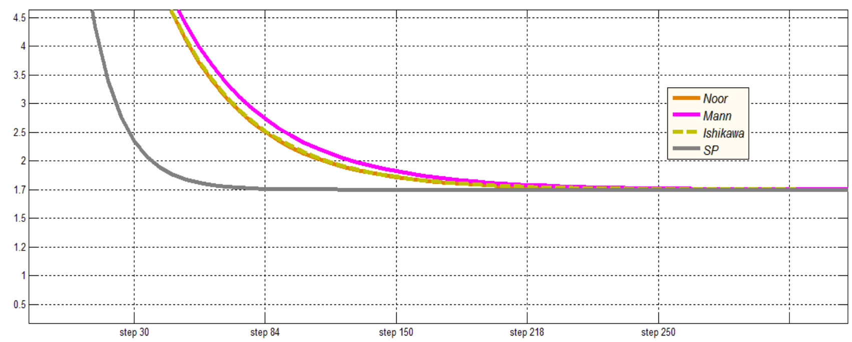

Figure 1. Comparison of rate of convergence among Noor, Mann, Ishikawa and SP

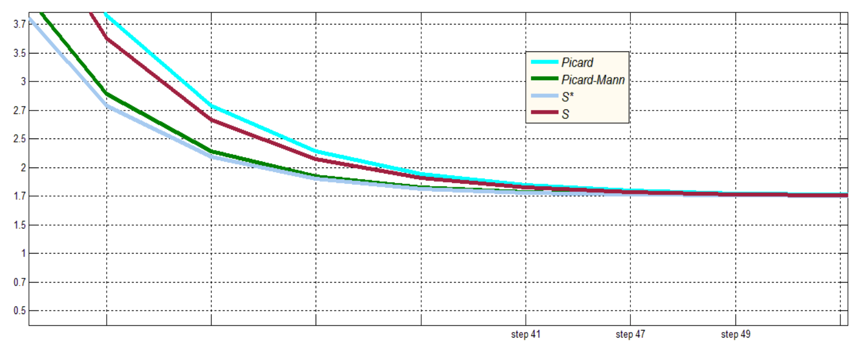

Figure 2. Comparison of rate of convergence among Picard, Picard-Mann,S* and S

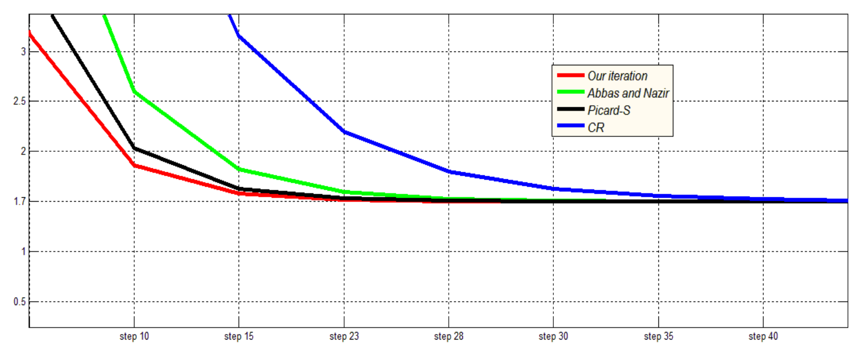

Figure 3. Comparison of rate of convergence among our iteration, Abbas-Nazir, Picard-S and CR

Example 2.6. Let $X = \mathbb{R}$ and $C = [0, \infty)$. Let $T : C \to C$ be a mapping defined by $T(x) = \sqrt{x^2 - 8x + 40}$ for all $x \in C$. It is easy to show that T has a unique fixed point $p_* = 5$. Choose $\alpha_n = 0.25$, $\beta_n = 0.40$, $\gamma_n = 0.70$ with the initial value $x_1 = 200$. The following tables show that the new iteration method (1.7) converges faster than all iteration methods which are mentioned in the previous example.

Table 4. Comparison rate of convergence among some iteration methods

x_n	Mann	Ishikawa	Noor	SP
x_1	200	200	200	200
x_2	199,0153037326100	198,6215491998820	198,3460095130540	194,6833352080640
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{63}	139,13287449695200	114,94795603465800	98,11575260075080	5,00000000000001
x_{64}	138,15506757872800	113,58617742269400	96,49055989897040	5,00000000000000
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{264}	5,00001351339154	5,00000000000246	5,00000000000000	5,00000000000000
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{290}	5,00000004084180	5,00000000000000	5,00000000000000	5,00000000000000
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{362}	5,00000000000000	5,00000000000000	5,00000000000000	5,00000000000000

Table 4 shows that SP iteration reaches the fixed point at the 64^{th} step while Noor, Ishikawa and Mann iterations reach 264^{th}step, 290^{th} step, and 362^{th} step, respectively.

Table 5. Comparison rate of convergence among some iteration methods

x_n	Picard	Picard-Mann	S^*	S
x_1	200	200	200	200
x_2	196,0612149304400	195,0768276600840	194,8012949067630	195,6674603977120
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{58}	5,00102675607005	5,00000000000175	5,00000000000003	5,00000023248493
x_{59}	5,00020545241574	5,00000000000028	5,00000000000000	5,00000004277723
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{62}	5,00000164382032	5,00000000000000	5,00000000000000	5,00000000026648
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{69}	5,00000000002104	5,00000000000000	5,00000000000000	5,00000000000000
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{75}	5,00000000000000	5,00000000000000	5,00000000000000	5,00000000000000

Table 5 shows that Picard iteration reaches the fixed point at the 75^{th} step while S, Picard-Mann and S* iterations reach at the 69^{th} step, 62^{th} step and 59^{th}step respectively.

Table 6. Comparison rate of convergence among some iteration methods

x_n	Our iteration	Picard-S	Abbas and Nazır	CR
x_1	200	200	200	200
x_2	191,1396240553450	191,7300586118630	191,5923857531320	193,9744119792280
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{34}	5,00000000000001	5,00000000000490	5,00000000000747	9,56932256205855
x_{35}	5,00000000000000	5,00000000000018	5,00000000000034	6,68283376485750
x_{36}	5,00000000000000	5,00000000000001	5,00000000000002	5,36905825343145
x_{37}	5,00000000000000	5,00000000000000	5,00000000000000	5,05357891261933
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{52}	5,00000000000000	5,00000000000000	5,00000000000000	5,00000000000000

Table 6 shows that our iteration reaches fixed point at the 35^{th} step, Picard-S and Abbas and Nazir iterations reach the fixed point at the 37^{th}step while CR iteration reaches the fixed point at the 52^{th} step.

Example 2.7. Let $X = \mathbb{R}$ and $C = [0, \infty)$. Let $T : C \to C$ be a mapping defined by $T(x) = x - 1 + \frac{1}{e^x}$ for all $x \in C$. It is easy to show that T has a unique fixed point $p_* = 0$. Choose $\alpha_n = \beta_n = \gamma_n = \frac{1}{9}$ with the initial value $x_1 = 3$. The following tables show that the new iteration method (1.7) converges faster than all iteration methods which are mentioned in the above examples.

Table 7. Comparison rate of convergence among some iteration methods

x_n	Mann	Ishikawa	Noor	SP
x_1	3	3	3	3
x_2	2,89442078537421	2,88330576108344	2,88213947357638	2,68517474137639
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{104}	0,00012335855652	0,00008877462777	0,00008681443723	0,000000000000000
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{305}	0,00000000000001	0,00000000000000	0,00000000000000	0,00000000000000
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_{328}	0,00000000000000	0,00000000000000	0,00000000000000	0,00000000000000

Table 7 shows that SP iteration reaches the fixed point at the 110^{th} step, Mann iteration reaches the fixed point at the 328^{th} and Noor and Ishikawa iterations reach the fixed point at the 324^{th} step, respectively.

Table 8. Comparison rate of convergence among some iteration methods

x_n	Picard	Picard-Mann	S^*	S
x_1	3	3	3	3
x_2	2,04978706836786	1,94975184975094	1,95190778766629	2,03867204407709
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_8	0,00000000007486	0,00000000000001	0,00000000000008	0,00000000003124
x_9	0,00000000000000	0,00000000000000	0,00000000000000	0,00000000000000

Table 8 shows that Picard, Picard-Mann, S^* and S iteration reach the fixed point at the 9^{th} step.

Table 9. Comparison rate of convergence among some iteration methods

x_n	Our iteration	Picard-S	Abbas and Nazır	CR
x_1	3	3	3	3
x_2	1,09483319871399	1,16887354205371	1,25563779846452	1,94202776607450
x_3	0,06726292068775	0,09644256312723	0,16086015122482	1,00321225851571
x_4	0,00000193275768	0,00000969017760	0,00116773275219	0,33108061715233
x_5	0,00000000000000	0,00000000000000	0,00000005984445	0,04304896429304
x_6	0,00000000000000	0,00000000000000	0,00000000000000	0,00079355690812
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_9	0,00000000000000	0,00000000000000	0,00000000000000	0,00000000000000

Table 9 shows that our iteration and Picard-S iteration reach the fixed point at the 5^{th} step, but the 4^{th} step shows that our iteration process is faster than Picard-S iteration method. Abbas and Nazir iteration reaches the fixed point at the 6^{th}step while CR iteration reaches the fixed point at the 9^{th} step.

Theorem 2.7. *Let S be an approximate operator of T. Let $\{x_n\}_{n=1}^{\infty}$ be an iterative sequence generated by (1.7) for T and define an iterative sequence $\{u_n\}_{n=1}^{\infty}$ as follows:*

$$\begin{cases} u_1 \in C, \\ u_{n+1} = Sv_n \\ v_n = (1-\alpha_n)\, w_n + \alpha_n S w_n \\ w_n = Su_n. \end{cases}$$

where $\{\alpha_n\}_{n=1}^{\infty}$ is a real sequence in $[0,1]$ satisfying $\frac{1}{2} \leq \alpha_n$ for all $n \in \mathbb{N}$. If $Tp_* = p_*$ and $Sx_* = x_*$ such that $u_n \to x_*$ as $n \to \infty$, then we have

$$\|p_* - x_*\| \leq \frac{5\varepsilon}{1-\delta},$$

where $\varepsilon > 0$ is a fixed number.

Proof. Let us consider the following iteration method defined by (1.7) according to S,

$$\begin{cases} u_0 \in C, \\ u_{n+1} = Sv_n \\ v_n = (1-\alpha_n)\, w_n + \alpha_n S w_n \\ w_n = Su_n \ (n \in \mathbb{N}). \end{cases} \tag{2.15}$$

From (1.7), (1.5) and (2.15), we have

$$\begin{aligned} \|z_n - w_n\| &= \|Tx_n - Su_n\| \le \|Tx_n - Tu_n\| + \|Tu_n - Su_n\| \\ &\le \delta \|x_n - u_n\| + L \|x_n - Tx_n\| + \varepsilon, \end{aligned} \tag{2.16}$$

and

$$\begin{aligned} \|y_n - v_n\| &= \|(1-\alpha_n)\, z_n + \alpha_n T z_n - (1-\alpha_n)\, w_n - \alpha_n S w_n\| \\ &\le (1-\alpha_n)\|z_n - w_n\| + \alpha_n \|Tz_n - Sw_n\| \\ &\le (1-\alpha_n)\|z_n - w_n\| + \alpha_n \|Tz_n - Tw_n\| \\ &\quad + \alpha_n \|Tw_n - Sw_n\| \\ &\le (1-\alpha_n)\|z_n - w_n\| + \alpha_n \delta \|z_n - w_n\| \\ &\quad + \alpha_n L \|z_n - Tz_n\| + \alpha_n \varepsilon \\ &= [1 - \alpha_n(1-\delta)]\|z_n - w_n\| + \alpha_n L \|z_n - Tz_n\| \\ &\quad + \alpha_n \varepsilon. \end{aligned} \tag{2.17}$$

Substituting (2.16) in (2.17), we obtain

$$\begin{aligned} \|y_n - v_n\| &\le [1 - \alpha_n(1-\delta)]\{\delta \|x_n - u_n\| + L\|x_n - Tx_n\| + \varepsilon\} \\ &\quad + \alpha_n L \|z_n - Tz_n\| + \alpha_n \varepsilon, \end{aligned} \tag{2.18}$$

and

$$\begin{aligned} \|x_{n+1} - u_{n+1}\| &= \|Ty_n - Sv_n\| \\ &\le \|Ty_n - Tv_n\| + \|Tv_n - Sv_n\| \\ &\le \delta \|y_n - v_n\| + L\|y_n - Ty_n\| + \varepsilon. \end{aligned} \tag{2.19}$$

Substituting (2.18) in (2.19), we obtain

$$\begin{aligned} \|x_{n+1} - u_{n+1}\| &\le \delta[1 - \alpha_n(1-\delta)]\{\delta\|x_n - u_n\| + L\|x_n - Tx_n\| + \varepsilon\} \\ &\quad + \alpha_n \delta L \|z_n - Tz_n\| + \alpha_n \delta \varepsilon + L\|y_n - Ty_n\| + \varepsilon \\ &= \delta^2[1 - \alpha_n(1-\delta)]\|x_n - u_n\| \\ &\quad + \delta[1 - \alpha_n(1-\delta)]L\|x_n - Tx_n\| \\ &\quad + \delta[1 - \alpha_n(1-\delta)]\varepsilon + \alpha_n \delta L\|z_n - Tz_n\| \\ &\quad + \alpha_n \delta \varepsilon + L\|y_n - Ty_n\| + \varepsilon. \end{aligned} \tag{2.20}$$

Since $\delta \in (0,1)$ and $\alpha_n \in [0,1]$ for all $n \in \mathbb{N}$ we have

$$\begin{aligned} 1-\alpha_n(1-\delta) &< 1, \\ \delta[1-\alpha_n(1-\delta)] &< 1, \\ (1-\alpha_n) &< 1, \\ \alpha_n\delta &< 1, \end{aligned}$$

and using hypothesis, we obtain

$$1-\alpha_n \le \alpha_n.$$

Hence, from (2.20) and the above inequalities, we have

$$\begin{aligned} \|x_{n+1}-u_{n+1}\| &\le [1-\alpha_n(1-\delta)]\,\|x_n-u_n\| + \alpha_n\delta(1+\delta)L\,\|x_n-Tx_n\| \\ &\quad +2\alpha_n\varepsilon + \delta\alpha_n L\,\|z_n-Tz_n\| + \alpha_n\varepsilon \\ &\quad +2\alpha_n L\,\|y_n-Ty_n\| + 2\alpha_n\varepsilon \\ &= [1-\alpha_n(1-\delta)]\,\|x_n-u_n\| \\ &\quad +\alpha_n(1-\delta)\left\{\frac{\left\{\begin{array}{c}\delta(1+\delta)L\,\|x_n-Tx_n\|+2L\,\|y_n-Ty_n\| \\ +\delta L\,\|z_n-Tz_n\|+5\varepsilon\end{array}\right\}}{(1-\delta)}\right\} \end{aligned}$$

Denote that

$$\begin{aligned} a_n &= \|x_n-u_n\| \\ \mu_n &= \alpha_n(1-\delta)\in(0,1) \\ \eta_n &= \left\{\frac{\left\{\begin{array}{c}\delta(1+\delta)L\,\|x_n-Tx_n\|+2L\,\|y_n-Ty_n\| \\ +\delta L\,\|z_n-Tz_n\|+5\varepsilon\end{array}\right\}}{(1-\delta)}\right\}. \end{aligned}$$

It follows from Lemma 1.1 that

$$\begin{aligned} 0 &\le \lim_{n\to\infty}\sup\|x_n-u_n\| \\ &\le \lim_{n\to\infty}\sup\left\{\frac{\left\{\begin{array}{c}\delta(1+\delta)L\,\|x_n-Tx_n\|+2L\,\|y_n-Ty_n\| \\ +\delta L\,\|z_n-Tz_n\|+5\varepsilon\end{array}\right\}}{(1-\delta)}\right\} \\ &= \frac{5\varepsilon}{(1-\delta)}. \end{aligned}$$

We know from Theorem 2.1 that $x_n \to p_*$ and using hypothesis, we obtain

$$\|p_*-x_*\| \le \frac{5\varepsilon}{1-\delta}.$$

3. Conclusion

After comparing the new iteration defined in this paper with the aforementioned iterations, we conclude that (1.7) is the fastest one among three step iteration methods in current literature.

References

[1] M. Abbas, T.Nazir, *A new faster iteration process applied to constrained minimization and feasibility problems*, Matematicki Vesnik, **66**(2014), 223.

[2] R.P. Agarwal, D. O'Regan, D.R. Sahu, *Iterative construction of fixed points of nearly asymptotically nonexpansive mappings*, J. Nonlinear Convex Anal., **8**(2007), 61.

[3] V. Berinde, *Iterative Approximation of Fixed Points*, Springer, Berlin, 2007.

[4] V. Berinde, *On the approximation of fixed points of weak contractive mappings*, Carpathian J. Math., **19**(2003), 7-22.

[5] R. Chugh, V. Kumar, S. Kumar, *Strong Convergence of a New Three Step Iterative Scheme in Banach Spaces*, Amer. J. Computational Math., **2**(2012), 345-357.

[6] K. Dogan, V. Karakaya, *On the convergence and stability results for a new general iterative process*, The Scientific World J., **2014**(2014), 8 pages.

[7] F. Gürsoy, V. Karakaya, B.E. Rhoades, *Data dependence results of new multi-step and S-iterative schemes for contractive-like operators*, Fixed Point Theory Appl., **2013**(2013).

[8] F. Gürsoy, V. Karakaya, *A Picard-S hybrid type iteration method for solving a differential equation with retarded argument*, arXiv preprint arXiv:1403.2546, (2014).

[9] S. Ishikawa, *Fixed Point By a New Iteration Method*, Proc. Amer. Math. Soc., **44**(1974), 147-150.

[10] I. Karahan, M. Özdemir, *A general iterative method for approximation of fixed points and their applications*, Advances in Fixed Point Theory, **3**(2013), 510-526.

[11] V. Karakaya, K. Doğan, F. Gürsoy, M. Ertürk, *Fixed Point of a New Three-Step Iteration Algorithm under Contractive-Like Operators over Normed Spaces*, Abstract Appl. Anal., **2013**(2013), 9 pages.

[12] W.R. Mann, *Mean Value Methods in Iteration*, Proc. Amer. Math. Soc., **4**(1953), 506-510.

[13] M.A. Noor, *New Approximation Schemes for General Variational Inequalities*, J. Math. Anal. Appl., **251**(2000), 217-229.

[14] M.O. Osilike, *Stability results for the Ishikawa fixed point iteration procedure*, Indian J. Pure Appl. Math., **26**(1995), 937–945.

[15] W. Pheungrattana, S. Suantai, *On the Rate of Convergence of Mann, Ishikawa, Noor and SP Iterations for Continuous on an Arbitrary Interval*, J. Comput. Appl. Math., **235**(2011), 3006-3914.

[16] E. Picard, *Memoire sur la theorie des equations aux derivees partielles et la methode des approximations successives*, J. Math. Pures Appl., **6**(1890), 145-210.

[17] S.H. Khan, *A Picard-Mann hybrid iterative process*, Fixed Point Theory Appl., **1**(2013), 1-10.

[18] S.M. Şoltuz, T. Grosan, *Data dependence for Ishikawa iteration when dealing with contractive like operators*, Fixed Point Theory Appl., **2008**(2008), 7 pages.

[19] X. Weng, *Fixed point iteration for local strictly pseudocontractive mapping*, Proc. Amer. Math. Soc., **113**(1991), 727-731.

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