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A new heavy-tailed distribution defined on the bounded interval: the logit slash distribution and its application

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ABSTRACT

This paper proposes a new heavy-tailed and alternative slash type distribution on a bounded interval via a relation of a slash random variable with respect to the standard logistic function to model the real data set with skewed and high kurtosis which includes the outlier observation. Some basic statistical properties of the newly defined distribution are studied. We derive the maximum likelihood, least-square, and weighted least-square estimations of its parameters. We assess the performance of the estimators of these estimation methods by the simulation study. Moreover, an application to real data demonstrates that the proposed distribution can provide a better fit than well-known bounded distributions in the literature when the skewed data set with high kurtosis contains the outlier observations.

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Logit slash distribution; unit slash distribution; heavy tailed; logistic function; outlier

1. Introduction

The bounded distributions, defined on (0,1) interval, have been applied to model the behavior of random variables limited to intervals of (0,1) length. These distributions have found applications in fields like meteorology, medicine, biology, hydrology, economics, actuarial, ecology, forestry, lifetime, financial modeling, and other sciences. No doubt, the beta distribution is one the first distribution that comes to mind to model the percentages and proportions. To introduce more flexible distribution than beta distribution there are some alternative distributions that have been proposed in distribution literature such as the Topp–Leone [43], Kumaraswamy (Kw) [28], arcsine [3], generalized beta type [33], standard two-sided power [44], Mc Donald arcsine (McA) [11], exponentiated Topp–Leone (ETL) [37], two-sided generalized Kumaraswamy [26], generalized beta [20] and unit Lindley distributions [30].

In addition, the distributions defined on (0,1) interval have been presented via some transformations of the random variable (rv). For example, following transformation related to logistic function structure

$$W = \frac{e^{(Z-\mu)/\sigma}}{1 + e^{(Z-\mu)/\sigma}} = \frac{1}{1 + e^{-(Z-\mu)/\sigma}} = 1 - \frac{1}{1 + e^{(Z-\mu)/\sigma}} \quad (1)$$

can be used to obtain bounded distribution on $(0,1)$, where Z is rv defined on $(-\infty, \infty)$ interval, $\mu \in \Re$ is location parameter and $\sigma > 0$ is the scale parameter. This logistic function is known as the inverse function of logit function. If Z rv has standard normal rv in (1), then W rv is named as Johnson S_B distribution, which is introduced by Johnson [23], with the following probability density function (pdf),

$$f(w, \mu, \sigma) = \frac{\sigma}{w(1-w)} \phi \left(\sigma \log \left(\frac{w}{1-w} \right) + \mu \right), \quad 0 < w < 1,$$

where $\phi(\cdot)$ is the pdf of the standard normal distribution. We denote it with Johnson $S_B(\mu, \sigma)$. In fact that the logit normal distribution, $LN(\mu, \sigma)$, is obtained for Johnson $S_B(-\mu/\sigma, 1/\sigma)$. One may see Mead [32], Frederic and Lad [13] and Pinson [36] for LN distribution. If Z rv has standard Laplace and standard logistic rv in (1), then W rv is named as Johnson S'_B distribution, which is introduced by Johnson [24], and unit logistic distribution, which is introduced by Tadikamalla and Johnson [42], respectively.

The distributions defined on $(0,1)$ interval can be generated by the logarithmic transformation of the type $W = e^{-Z}$, where Z is rv defined on $(0, \infty)$ interval. The unit-gamma [10], the log-Lindley [19], log-xgamma [2], unit Birnbaum-Saunders [31], unit inverse Gaussian (UIG) [17] and log-weighted exponential [1] distributions can be given as some examples of this transformation. Examples of the bounded distributions are also derived from the transformation of the type $W = Z/(1+Z)$, where Z is rv defined on $(0, \infty)$ interval. The unit Lindley distribution is an example of this method. The transformation methods can be increased by other functions. These transformations have generated bounded distributions, which are more flexible than beta distribution in terms of data modeling.

On the other hand, slash type (scale mixture type) distributions are well-known as heavy-tailed or thick-tailed distributions. When data set has the outlier observation, heavy-tailed distributions have been proposed by many statisticians to increase efficiency of the inferences based on data set. The t distribution is a good alternative to normal distribution since the normal distribution is sensitive to outlier observation. Another popular alternative heavy-tailed distribution is the ordinary slash distribution which has been introduced by [41] with the following stochastic representation

$$Y = \mu + \sigma \frac{Z}{U^{1/q}}, \quad (2)$$

where $q > 0$ is the shape parameter, which controls the tail thickness and kurtosis of the distribution, $\mu \in \Re$ is location parameter, $\sigma > 0$ is the scale parameter and standard normal rv Z is independently distributed of the uniform rv U on $(0,1)$. The pdf and cumulative distribution function (cdf) of the slash distribution are respectively given by

$$f_{Sl}(y, q, \mu, \sigma) = \frac{q}{\sigma} \int_0^1 t^q \phi \left(t \left[\frac{y - \mu}{\sigma} \right] \right) dt, \quad y \in \Re \quad (3)$$

and

$$F_{Sl}(y, q, \mu, \sigma) = q \int_0^1 t^{q-1} \Phi \left(t \left[\frac{y - \mu}{\sigma} \right] \right) dt, \quad y \in \Re,$$

where $\Phi(\cdot)$ is the cdf of standard normal distribution. We denote it with $Sl(q, \mu, \sigma)$. The standard slash distribution or canonical slash is obtained for the $Sl(1, \mu, \sigma)$. The normal

distribution is obtained as $q \rightarrow \infty$. The slash distribution has heavier tails and larger kurtosis than the normal distribution as well as symmetric bell-shaped. The Sl distribution are also useful in robustness studies (see [22,25,34,41]). Further, the heavy-tailed distributions with slash type have been introduced in the literature such as multivariate skew slash [45], exponential power slash [15], another multivariate skew slash [4], multivariate symmetric slash [5], half normal slash [35], modified slash [38], beta slash [16], beta Moyal slash [14], matrix variate multivariate slash [9], Rayleigh slash [21], slashed half t [8], gamma slash [27], Lindley–Weibull slash [39] and Gumbel slash [18] distributions. These slash type distributions have supplied nice results on data modeling.

The first objective of this paper is to introduce a new distribution on $(0,1)$ interval. We define it by making use of the idea lying in the transformation of the logistic function which is given by (1). In this idea, Z rv will be a slash rv, which is pointed out by (2). To the best of our knowledge, there is no bounded slash type distribution on $(0,1)$ interval except the truncated slash distribution. The second objective of the paper is to obtain an alternative bounded distribution for modeling the data sets involving asymmetric, heavy tails and outliers. The newly defined distribution will have heavier tails than the Johnson S_B distribution, and it will be more useful for modeling data sets involving asymmetric, heavy tails, and outliers. Therefore, it will be naturally a robust alternative to the Johnson S_B distribution. The newly defined distribution is named as logit slash distribution, and its statistical properties have been studied. We consider the maximum likelihood estimation, least-square estimation, and weighted least-square estimation procedures to estimate of the model parameters and give a simulation study to see performances of these estimation methods. To illustrate its applicability on real phenomena, an application of the model to a real data set with skewed and high kurtosis, which includes in the outlier observations, is presented and compared to the fit attained by some other well-known distributions on the $(0,1)$ interval. The paper is ended with future work and conclusion remarks.

2. Logit slash distribution

To introduce a new heavy-tailed bounded distribution on $(0,1)$, we consider the following definition.

Definition 2.1: A random variable X has a logit slash distribution with shape parameter q , location parameter μ and scale parameter σ if its pdf is given by

$$f(x, q, \mu, \sigma) = \frac{q}{x(1-x)\sigma} \int_0^1 t^q \phi \left(t \left[\frac{\log \left(\frac{x}{1-x} \right) - \mu}{\sigma} \right] \right) dt, \quad (4)$$

for $0 < x < 1$, $q, \sigma > 0$ and $-\infty < \mu < \infty$.

Proof: Define $X = e^Y / (1 + e^Y)$ random variable, where Y has slash random variable with pdf (3). Then, we obtain $g^{-1}(x) = \log(x/(1-x))$ and $\partial g^{-1}(x) / \partial x = 1/(x(1-x))$. Now, we use the chain rule to compute the density of X

$$f(x, q, \mu, \sigma) = f_{Sl}(g^{-1}(x), q, \mu, \sigma) \left| \frac{\partial g^{-1}(x)}{\partial x} \right|.$$

Hence, the proof is completed. ■

On the other word, a random variable X is distributed the logit slash distribution on the interval $(0,1)$ if its logit transformation, $\log(X/(1-X))$, is distributed $Sl(q, \mu, \sigma)$. Therefore, we call it the logit slash distribution. We denote it with $LSl(q, \mu, \sigma)$. Since the pdf of the LSl distribution is reduced to pdf of the Johnson S_B distribution as $q \rightarrow \infty$, we generalize the Johnson S_B distribution. The LSl distribution is also defined by the following hierarchical model

$$\begin{aligned} X|U &\sim LN(\mu, \sigma U^{-1/q}) \\ U &\sim Uniform(0, 1) \\ X &\sim LSl(q, \mu, \sigma). \end{aligned}$$

This compounding is given by the following pdf formula

$$\begin{aligned} f(x, q, \mu, \sigma) &= \int_0^1 f_{X|U}(x|u) f_U(u) du \\ &= \int_0^1 \frac{u^{1/q}}{x(1-x)\sigma} \phi\left(u^{1/q} \left[\frac{\log\left(\frac{x}{1-x}\right) - \mu}{\sigma} \right]\right) du. \end{aligned}$$

We obtain Equation (4) by transforming the variable into $t = u^{1/q}$.

The cdf of the LSl distribution is given as follows:

$$\begin{aligned} F(x, q, \mu, \sigma) &= \frac{q}{\sigma} \int_0^1 t^q \int_0^x [w(1-w)]^{-1} \phi\left(t \left[\frac{\log\left(\frac{w}{1-w}\right) - \mu}{\sigma} \right]\right) dw dt, \\ \omega &= t \left[\frac{\log\left(\frac{w}{1-w}\right) - \mu}{\sigma} \right] \\ &= q \int_0^1 t^{q-1} \left[\int_{-\infty}^{\left[\frac{\log\left(\frac{x}{1-x}\right) - \mu}{\sigma} \right] t} \phi(\omega) d\omega \right] dt \\ &= q \int_0^1 t^{q-1} \Phi\left(t \left[\frac{\log\left(\frac{x}{1-x}\right) - \mu}{\sigma} \right]\right) dt, \quad 0 < x < 1, \end{aligned} \quad (5)$$

where $\Phi(\cdot)$ is the cdf of the standard normal distribution, that is $F(x, q, \mu, \sigma) = F_{Sl}(\log[x/(1-x)], q, \mu, \sigma)$. Now, we give its some distributional properties with the following subsections.

2.1. Density shape

Since the LSl distribution has three parameters, which control tail thickness, kurtosis, skewness and pdf shapes of distribution, we can obtain bounded slash distributions on interval $(0,1)$ with very flexible forms. The distribution is symmetric about $\frac{1}{2}$ when $\mu = 0$. This

situation is given as follows.

$$f\left(\frac{1}{2} - x, q, 0, \sigma\right) = \frac{q}{\left(\frac{1}{2} - x\right)\left(\frac{1}{2} + x\right)\sqrt{2\pi}\sigma} \int_0^1 t^q \times \exp\left\{\frac{-t^2}{2} \left(\frac{\log\left(\frac{1}{2} - x\right) - \log\left(\frac{1}{2} + x\right)}{\sigma}\right)^2\right\} dt$$

and

$$f\left(\frac{1}{2} + x, q, 0, \sigma\right) = \frac{q}{\left(\frac{1}{2} + x\right)\left(\frac{1}{2} - x\right)\sqrt{2\pi}\sigma} \int_0^1 t^q \times \exp\left\{\frac{-t^2}{2} \left(\frac{\log\left(\frac{1}{2} + x\right) - \log\left(\frac{1}{2} - x\right)}{\sigma}\right)^2\right\} dt.$$

It is clear that $f(1/2 - x, q, 0, \sigma) = f(1/2 + x, q, 0, \sigma)$. This means that the density function is symmetric, if $\mu = 0$, in which case expected value of distribution is equal to $1/2$ (see Figure 1). In more general terms, $f(x, q, \mu, \sigma) = f(1 - x, q, -\mu, \sigma)$ is holded. Hence, if $LSl(q, -\mu, \sigma)$ is right skewed, the $LSl(q, \mu, \sigma)$ is just left skewed.

We sketched the plots of the pdf to see its possible density shapes. From Figure 1, we see that pdf shapes of the LSl distribution can be w-shaped, U-shaped, bi-modal shaped, bell (uni-modal) shaped, N-shaped, decreasing and increasing. It is well known that the ordinary slash distribution has a thicker tail than the ordinary normal distribution due to its q shape parameter which controls tail thickness distribution. Thus, we expect that the LSl distribution also will have thicker tail than the logit normal distribution. As it can be seen that for the same $\mu = 0$ and $\sigma = 1$ parameters values, this situation is valid (see Figure 1).

Further, Figure 2 indicates the shape regions of pdf of the LSl distribution for fixed $\mu = 0$ and $q = 15$. As seen from this Figure, possible eight shape regions of pdf have been scanned via changing parameters for fixed μ and q . When the LSl distribution is symmetric about $\frac{1}{2}$, shapes of distribution can be U-shaped or w-shaped for changing parameters q and σ . When $q = 15$, the shapes of distribution can be bi-modal or uni-modal for changing parameters μ and σ .

Consequently, we can say that this new model can be more useful for various data set than other bounded models. All in all, the shapes of this distribution can be a distinguishing feature on data modeling.

2.2. Mode of distribution

Mode(s) of distribution is(are) known as the value which maximize(s) its density at local maxima point(s). At the same time, density can be minimum value(s) at the antimode(s) where is (are) local minima point(s). These appear as distinct extreme points (local maxima or local minima points) in the density function. At the local minima points, U-shaped and w-shaped distributions have uni-antimode and bi-antimode respectively. We note that there is a mode between two local minima points for the w-shaped distributions.

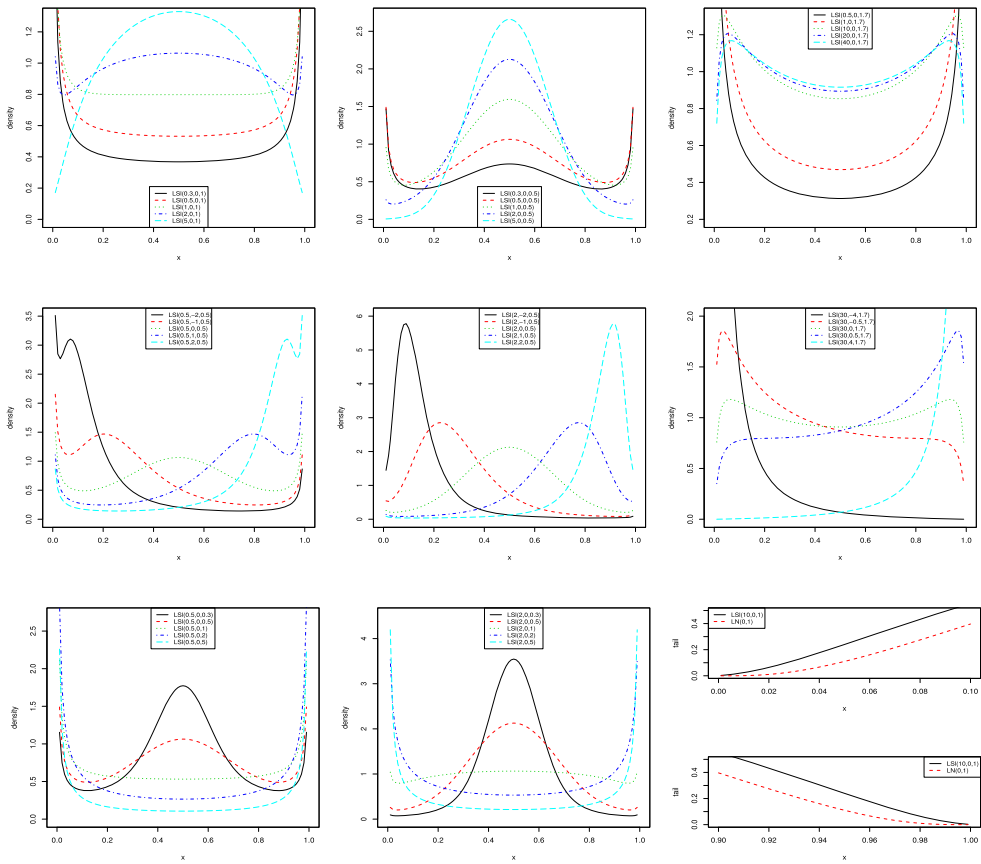


Figure 1. The possible pdf shapes and tail plot of the *LSI* distribution for selected parameters values.

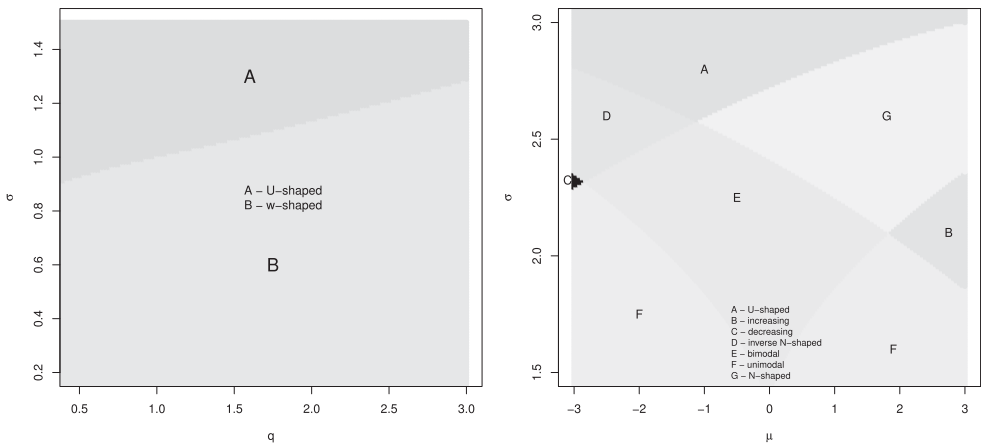


Figure 2. The possible pdf shape regions of the *LSI* distribution for fixed $\mu = 0$ (left) and $q = 15$ (right) parameters values.

Table 1. Some mode and antimode values of the *LSI* distribution for selected parameters.

(q, μ, σ)	Mode point(s)	Value(s) at mode	Antimode point(s)	Value(s) at antimode	Density shape
(0.5,0,1)	—	—	0.5	0.5319	U-shaped
(1,0,1)	—	—	0.5	0.7978	U-shaped
(2,3,2)	—	—	0.2715	0.2347	U-shaped
(0.3,0,0.5)	0.5*	0.7365**	0.1415 (0.8585)	0.4038	w-shaped
(0.3,1,0.5)	0.8055*	1.0277**	0.2161 (0.9228)	0.2125 (0.8853)	w-shaped
(0.3,—,1,0.5)	0.1945*	1.0277**	0.7839 (0.0772)	0.2125 (0.8853)	w-shaped
(0.5,—0.5,0.45)	0.3429*	1.2806**	0.0892 (0.6891)	0.8434 (0.3235)	w-shaped
(15,2,2)	0.9987	13.6627	—	—	uni-modal
(15,—2,2)	0.0013	13.6627	—	—	uni-modal
(15,0,1)	0.5	1.4960	—	—	uni-modal
(30,0,1.7)	0.0628 (0.9372)	1.1801	—	—	bi-modal
(12,0,1,1.65)	0.0612 (0.9542)	1.1026 (1.3185)	—	—	bi-modal
(12,—0,1,1.65)	0.0458 (0.9388)	1.3185 (1.1026)	—	—	bi-modal

*Mode between two local minima points.
 **Densitiy value at mode between two local minima points.

Differentiating (1) and equating to zero, we have

$$\begin{aligned}
 &x^2\sigma^3\left(1-x\right)f\left(x,q,\mu,\sigma\right)-x\sigma^3\left(1-x\right)^2f\left(x,q,\mu,\sigma\right)-q\left(\log\left(\frac{x}{1-x}\right)-\mu\right) \\
 &\times\int_0^1t^{q+2}\phi\left(t\left[\frac{\log\left(\frac{x}{1-x}\right)-\mu}{\sigma}\right]\right)dt=0.
 \end{aligned}$$

Since the pdf of the *LSI* distribution contains the integral form, its first derivation contains the integral form naturally. Therefore, its mode(s) or antimode(s) can not be obtained explicitly. However, we numerically calculated its modes and antimodes using R program which includes the one-dimensional optimization method that is called the Brent method [7]. We obtain some mode points and its values at mode point for selected parameters values given by Table 1.

2.3. Hazard rate function

The hazard rate function (hrf), failure rate function, is an important tool in reliability analysis. On the support of the *LSI* distribution, its hrf is given by

$$h\left(x,q,\mu,\sigma\right)=\frac{q}{x\left(1-x\right)\sigma}\frac{\int_0^1t^q\phi\left(t\left[\frac{\log\left(\frac{x}{1-x}\right)-\mu}{\sigma}\right]\right)dt}{\left(1-q\int_0^1t^{q-1}\Phi\left(t\left[\frac{\log\left(\frac{x}{1-x}\right)-\mu}{\sigma}\right]\right)dt\right)},\quad 0<x<1. \tag{6}$$

Analytically, it can be difficult to identify the hrf shapes of distribution. That’s why we sketched the plot of hrf to see its possible shapes. From Figure 3, we see that hrf shapes of the *LSI* distribution can be an increasing function as monotone shaped and can be non-monotone shaped such as bathtub shaped, N-shaped (modified bathtub shaped), and w-shaped. According to Bebbington *et al.* [6], N-shaped hrf can appear in mortality among breast cancer patients, and w-shaped hrf can be seen in mixing of two or more lifetime distributions as well as these hrf shapes can be obtained using polynomial functions of degree

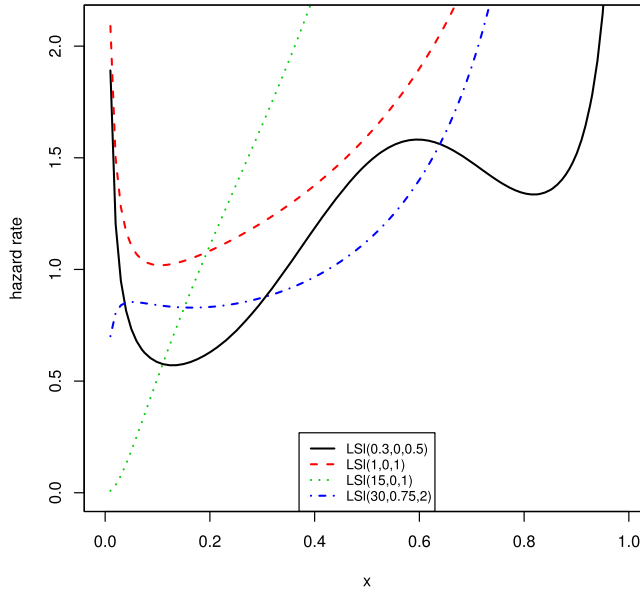


Figure 3. The possible hrf shape of the *LSI* distributions for selected parameters values.

three or four. It is a striking property that the *LSI* distribution has N-shaped and w-shaped hrf on a bounded interval (0,1). Thus, hrf shapes of this distribution can be distinguishing feature on data modeling as well as in its density. The beta and Kw distributions have not these hrf shapes. Further, Figure 4 indicates the possible shape regions of hrf of the *LSI* distribution for fixed $\mu = 0$ and $q = 2$. As seen from Figure 4, four shape regions of hrf have been scanned via changing parameters for fixed μ and q .

2.4. Moments and related measures

When $X \sim LSI(q, \mu, \sigma)$, the r^{th} raw moment of X is given by

$$\begin{aligned}
 \mu'_r &= \int_0^1 x^r f(x, q, \mu, \sigma) dx, \quad \omega = t \left[\frac{\log\left(\frac{x}{1-x}\right) - \mu}{\sigma} \right] \\
 &= q \int_0^1 t^{q-1} \int_{-\infty}^{+\infty} \left(1 - \frac{1}{1 + e^{\mu + \omega\sigma/t}} \right)^r \phi(\omega) d\omega dt \\
 &= \frac{q}{\sqrt{2\pi}} \sum_{i=0}^r (-1)^i \binom{r}{i} \int_0^1 t^{q-1} \int_{-\infty}^{+\infty} e^{-\omega^2/2} (1 + e^{\mu + \omega\sigma/t})^{-i} d\omega dt \\
 &= 1 + \frac{q}{\sqrt{2\pi}} \sum_{i=1}^r (-1)^i \binom{r}{i} \int_0^1 t^{q-1} \int_{-\infty}^{+\infty} e^{-\omega^2/2} (1 + e^{\mu + \omega\sigma/t})^{-i} d\omega dt. \quad (7)
 \end{aligned}$$

Especially, the expected value, $E(X)$, is given by

$$\mu'_1 = 1 - \frac{q}{\sqrt{2\pi}} \int_0^1 t^{q-1} \int_{-\infty}^{+\infty} e^{-\omega^2/2} (1 + e^{\mu + \omega\sigma/t})^{-1} d\omega dt.$$

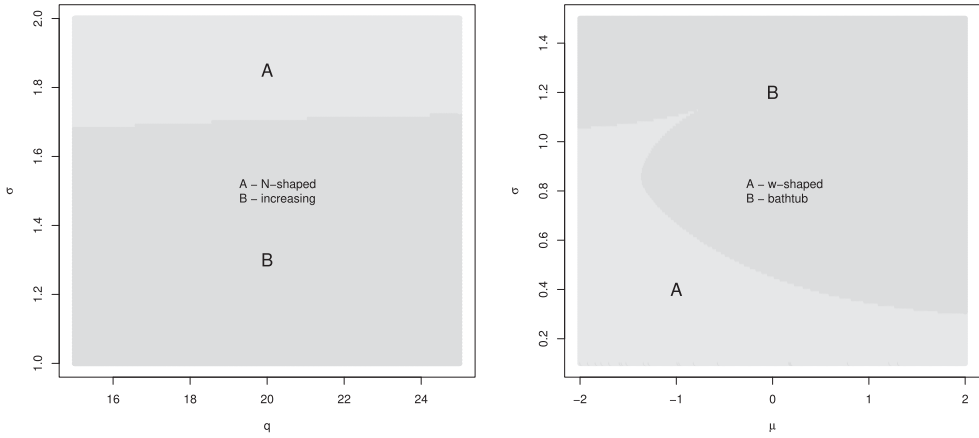


Figure 4. The possible hrf shapes regions of the *LSI* distribution for fixed $\mu = 0$ (left) and $q = 2$ (right) parameters values.

If we re-write (7), the following equation is obtained for μ'_r

$$\mu'_r = \frac{q}{\sqrt{2\pi}} \int_0^1 t^{q-1} \int_{-\infty}^{+\infty} \left(1 + e^{-(\mu + \omega\sigma/t)}\right)^{-r} e^{-\omega^2/2} d\omega dt. \quad (8)$$

As it can be seen, μ'_r 's can not be expressed in a closed form. The numerical integration can be applied to obtain the mean and other important related measures. We note that the integral, which related to ω in the (8), is the r^{th} raw moment of the Johnson S_B distribution. Hence, we can obtain alternative formula for μ'_r . Following results of the Johnson [23], the expected value of the *LSI* distribution is obtained by

$$\begin{aligned} \mu'_1 = & \frac{q}{\sqrt{2\pi}} \int_0^1 t^{q-1} e^{-(1/2)\left(\frac{t\mu}{\sigma}\right)^2} \left[\frac{\frac{\sigma}{2t} + \frac{\sigma}{t} \sum_{n=1}^{\infty} e^{-(\sigma n/t\sqrt{2})^2} \cosh \frac{n(\sigma^2/t^2 + 2\mu)}{2} \operatorname{sech} \left(\frac{n\sigma^2}{t^2} \right)}{1 + 2 \sum_{n=1}^{\infty} e^{-2(n\pi t/\sigma)^2} \cos(2\pi n\mu t^2/\sigma^2)} \right. \\ & \left. + \frac{\frac{t2\pi}{\sigma} \sum_{n=1}^{\infty} e^{-((2n-1)\pi t/\sigma\sqrt{2})^2} \sin[(2n-1)\mu t^2/\sigma^2] \operatorname{cosech} \left(\frac{\pi t\sqrt{(2n-1)}}{\sigma} \right)^2}{1 + 2 \sum_{n=1}^{\infty} e^{-2(n\pi t/\sigma)^2} \cos(2\pi n\mu t^2/\sigma^2)} \right] dt. \end{aligned} \quad (9)$$

The other moments can be given by partial derivatives of (8) respect to μ based on the recurrence formula. This procedure is given by followings based on (8).

$$\begin{aligned} \frac{\partial \mu'_r}{\partial \mu} &= r \frac{q}{\sqrt{2\pi}} \int_0^1 t^{q-1} \left[\int_{-\infty}^{+\infty} e^{-\omega^2/2} \left(1 + e^{-(\mu + \omega\sigma/t)}\right)^{-r-1} e^{-(\mu + \omega\sigma/t)} d\omega \right] dt \\ &= r \frac{q}{\sqrt{2\pi}} \int_0^1 t^{q-1} \left[\int_{-\infty}^{+\infty} e^{-\omega^2/2} \left(1 + e^{-(\mu + \omega\sigma/t)}\right)^{-r-1} \right. \\ &\quad \times \left[\left(1 + e^{-(\mu + \omega\sigma/t)}\right) - 1 \right] d\omega \left. \right] dt \\ &= r(\mu'_r - \mu'_{r+1}). \end{aligned}$$

Hence, we have

$$\mu'_{r+1} = \mu'_r - \frac{1}{r} \frac{\partial \mu'_r}{\partial \mu}. \quad (10)$$

Using (10), for $r = 1, 2, 3$ following equations are obtained by

$$\begin{aligned} \mu'_2 &= \mu'_1 - \frac{\partial \mu'_1}{\partial \mu}, \\ \mu'_3 &= \mu'_1 - \frac{3}{2} \frac{\partial \mu'_1}{\partial \mu} + \frac{1}{2} \frac{\partial^2 \mu'_1}{\partial \mu^2}, \\ \mu'_4 &= \mu'_1 - \frac{11}{6} \frac{\partial \mu'_1}{\partial \mu} + \frac{\partial^2 \mu'_1}{\partial \mu^2} - \frac{1}{6} \frac{\partial^3 \mu'_1}{\partial \mu^3}. \end{aligned}$$

The j th order central moment can be obtained by the following relationship

$$\mu_j = E[(X - \mu'_1)^j] = \sum_{r=0}^j \binom{j}{r} \mu'_r (-\mu'_1)^{j-r}, \quad j = 2, 3, \dots$$

With the above formula, the skewness and kurtosis coefficients are respectively given by

$$\sqrt{\beta_1} = \sqrt{\frac{\mu_3^2}{\mu_2^3}} \quad \text{and} \quad \beta_2 = \frac{\mu_4}{\mu_2^2}.$$

However, these above calculations can be easily computed using many packet programs such as R, Matlab, Maple, and Wolfram. It is noticed that it may be useful for the moment calculations of the *LSI* distribution to take $q > 2r$ since the moments of ordinary slash distribution are valid for $q > 2r$.

The plots of the $\sqrt{\beta_1}$ and β_2 for selected values of q , μ and σ are shown in Figure 5. From this Figure, we see that asymmetry and kurtosis of distribution depend on three parameters. For $\mu = 0$, the skewness of distribution is equal to zero as expected. When q increases, the positive (negative) skewness is seen for negative (positive) μ values. The skewness decreases for fixed q and σ , when μ increases. The negative (positive) skewness decreases for fixed $\mu > 0 (< 0)$ and σ , while q increases. When $\mu < 0$, kurtosis decreases. Otherwise kurtosis increases. When q increases, the kurtosis decreases for fixed μ . When σ increases, firstly skewness decreases then it increases for fixed q and μ . When q increases, firstly skewness increases then it decreases for fixed σ and μ . When σ increases, firstly kurtosis increases then it decreases for fixed q and μ . When q increases, firstly kurtosis decreases then it increases for fixed σ and μ . These plots indicate that the distribution can model various data types on the unit interval in terms of skewness and kurtosis.

2.5. Quantile function and random number generation

The *LSI* distribution can be simulated by inverting its cdf in (5) with the solution of the following non-linear equation:

$$\int_0^1 t^{q-1} \Phi \left(t \left[\frac{\log \left(\frac{x_u}{1-x_u} \right) - \mu}{\sigma} \right] \right) dt - \frac{u}{q} = 0,$$

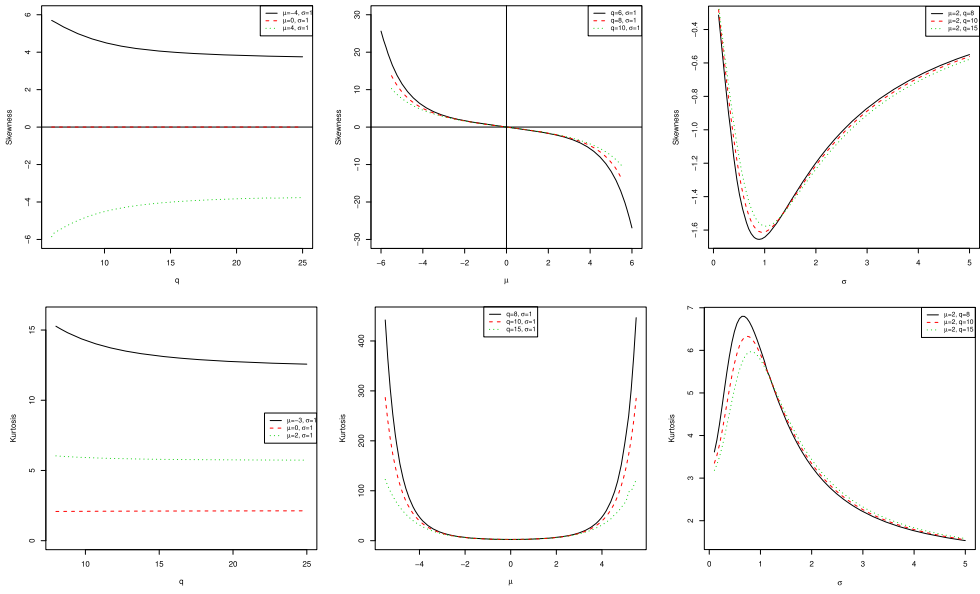


Figure 5. $\sqrt{\beta_1}$ and β_2 plots of the *LSI* distribution for selected parameters values.

where $u \sim \text{uniform}(0, 1)$ and x_u is the solution of the equation, that is u *th* quantile of the *LSI* distribution. Hence, if U is a uniform random variable on $(0, 1)$ then, X_U is the *LSI* random variable. The **uniroot** function of R software can be used to solve the above non-linear equation. In addition, we can obtain random number from the *LSI* distribution by the following algorithm. We can give this procedure as below.

Algorithm

- set q , μ and σ .
- simulate $U \sim \text{Uniform}(0, 1)$;
- simulate $Z \sim N(0, 1)$;
- compute $Y = \mu + \sigma Z / U^{1/q}$, then Y follows that $Sl(q, \mu, \sigma)$;
- compute $X = e^Y / (e^Y + 1)$, then X follows that $LSI(q, \mu, \sigma)$.

3. Estimation methods

In this section, we propose various estimators for the unknown parameters of the *LSI* distribution. We discuss the maximum likelihood, least squares and weighted least-squares estimation methods and compare their performances on the basis of simulated sample from the *LSI* distribution. We note that the components of the gradient (score) vectors which are belong to three estimation methods are given in Appendix.

3.1. Maximum likelihood estimation

In this subsection, we estimate the parameters of the *LSI* distribution by the method of maximum likelihood estimation (MLE). Let X_1, X_2, \dots, X_n be a random sample from the *LSI* distribution with observed values x_1, x_2, \dots, x_n and $\Xi = (q, \mu, \sigma)^T$ be the vector of the

model parameters. The log-likelihood function for Ξ may be expressed as

$$\ell(\Xi|x) = \ell = n \log q - n \log \sigma - \sum_{i=1}^n \log [x_i (1 - x_i)] + \sum_{i=1}^n \log \left[\int_0^1 t^q \phi(t u_i) dt \right], \quad (11)$$

where $u_i = (\log(x_i/(1 - x_i)) - \mu)/\sigma$ for $i = 1, 2, \dots, n$. We differentiate Equation (11) with respect to q , μ and σ to obtain the score vector ($U_q = \partial \ell / \partial q$, $U_\mu = \partial \ell / \partial \mu$, $U_\sigma = \partial \ell / \partial \sigma$)^T.

Setting $U_\lambda = U_q = U_\mu = U_\sigma = 0$ and solving them simultaneously, MLEs, say \hat{q} , $\hat{\mu}$ and $\hat{\sigma}$, are obtained.

For interval estimation of the parameters, we obtain the 3×3 observed information matrix $J_{rs} = J(\Xi) = \{\partial^2 \ell / \partial r \partial s\}$ (for $r, s = q, \mu, \sigma$), whose elements can be found from the author when needed. At the same time, these elements can be computed numerically by the packet program. Under standard regularity conditions when $n \rightarrow \infty$, the distribution of $\hat{\Xi}$ can be approximated by a multivariate normal $N_3(0, J(\hat{\Xi})^{-1})$ distribution to construct approximate confidence intervals for the parameters. Here, $J(\hat{\Xi})$ is the total observed information matrix evaluated at $\hat{\Xi}$. Then, approximate $100(1 - \delta)\%$ confidence intervals for q , μ and σ can be determined by: $\hat{q} \pm z_{\delta/2} \sqrt{\hat{J}_{qq}^{-1}}$, $\hat{\mu} \pm z_{\delta/2} \sqrt{\hat{J}_{\mu\mu}^{-1}}$ and $\hat{\sigma} \pm z_{\delta/2} \sqrt{\hat{J}_{\sigma\sigma}^{-1}}$, where $z_{\delta/2}$ is the upper δ th percentile of the standard normal model and \hat{J}_{ii}^{-1} are diagonal elements of $J(\hat{\Xi})^{-1}$ for $i = q, \mu$ and σ .

The likelihood ratio (LR) statistic can be used for comparing the *LSI* model with LN model (or Johnson S_B model). This comparison is to test equivalently H_0 : Model is LN ($q \rightarrow \infty$) versus H_1 : Model is *LSI*. For this situation, the LR statistic is computed with $w = 2[\ell(\hat{q}, \hat{\mu}, \hat{\sigma}) - \ell(\tilde{\mu}, \tilde{\sigma})]$, where $(\hat{q}, \hat{\mu}, \hat{\sigma})$ are the unrestricted MLEs and $(\tilde{\mu}, \tilde{\sigma})$ are the restricted estimates under H_0 . The statistic w is asymptotically (as $n \rightarrow \infty$) distributed as χ_ν^2 , where ν is the difference of two parameter vectors of nested models. For example, $\nu = 1$ for the above hypothesis test.

3.2. Least-squares estimation

Let $X_{(1)}, X_{(2)}, \dots, X_{(n)}$ be ordered statistics from the *LSI* distribution with sample size n . Then, the expectation of the empirical cumulative distribution function is defined as

$$E[F(x_{(i)})] = \frac{i}{n+1}; \quad i = 1, 2, \dots, n.$$

The least-square estimates (LSEs), say \hat{q}_{LSE} , $\hat{\mu}_{LSE}$ and $\hat{\sigma}_{LSE}$, of q , μ and σ are obtained by minimizing

$$QLSE(\Xi) = \sum_{i=1}^n \left(F(x_{(i)}, q, \mu, \sigma) - \frac{i}{n+1} \right)^2. \quad (12)$$

3.3. Weighted least-squares estimates

Let $X_{(1)}, X_{(2)}, \dots, X_{(n)}$ be ordered sample of size n from LSI distribution. The variance of the empirical cumulative distribution function is defined as

$$V[F(x_{(i)})] = \frac{i(n-i+1)}{(n+2)(n+1)^2}; i = 1, 2, \dots, n.$$

Then, the weighted least-square estimates (WLSEs) \hat{q}_{WLSE} , $\hat{\mu}_{WLSE}$ and $\hat{\sigma}_{WLSE}$ of q , μ and σ are obtained by minimizing

$$QWLSE(\Xi) = \sum_{i=1}^n \frac{\left(F(x_{(i)}, q, \mu, \sigma) - \frac{i}{n+1}\right)^2}{V[F(x_{(i)})]}. \quad (13)$$

Equations (11), (12) and (13) can be optimized directly by some well-known packet programs such as R (`optim` and `maxLik` routines), SAS (`PROC NLMIXED` routine), and Ox (`MaxBFGS` routine) to numerically optimize $\ell(\Xi)$, $QLSE(\Xi)$, and $QWLSE(\Xi)$ functions.

4. Simulation study

In this section, the performances of the MLEs, LSEs and WLSEs of the LSI distribution are discussed via a simulation study. The algorithm, which is given by Section 2.5, is used to generate random variables from the LSI distribution. We generate $N = 1000$ samples of size $n = 20, 30, \dots, 1000$ from the LSI distribution with true parameter values $q = 5$, $\mu = 0$, $\sigma = 1$. The performances of the above estimators is evaluated based on the empirical bias and mean square error (MSE) measurements. The empirical biases and MSEs are given by

$$\widehat{Bias}_{\epsilon}(n) = \frac{1}{N} \sum_{i=1}^N (\hat{\epsilon}_i - \epsilon)$$

and

$$\widehat{MSE}_{\epsilon}(n) = \frac{1}{N} \sum_{i=1}^N (\hat{\epsilon}_i - \epsilon)^2,$$

respectively, where $\epsilon = q, \mu, \sigma$. The results of above estimators were obtained by the `optim` function in the R program.

Figures 6–8 display the simulation results for the above measures. As seen from these figures, for three estimators, the empirical biases and MSEs approach zero when the sample size increases. Therefore, we can say that all estimators are to be consistent. Although the performances of the estimators are very close in all the cases for μ parameter, the performance of the MLE method is better than others in all the cases for q and σ parameters.

Furthermore, we also give a simulation study based on the above results of the MLEs to see the performance of the 95% confidence intervals. The performance of the MLEs is evaluated based on the average length (AL). The standard errors of the MLEs, namely

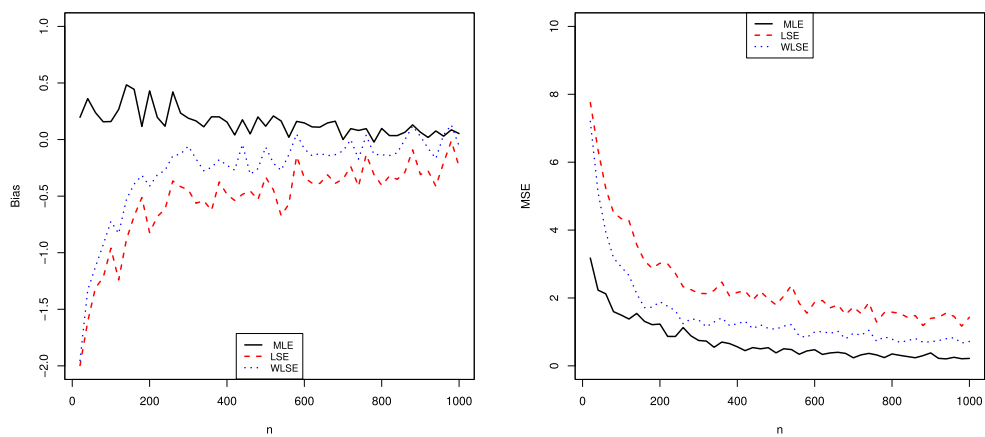


Figure 6. The simulation results of the q parameters.

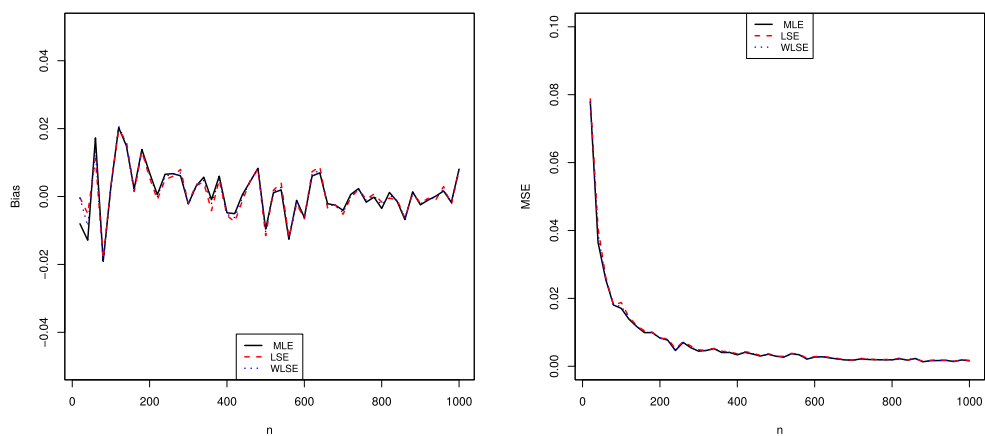


Figure 7. The simulation results of the μ parameters.

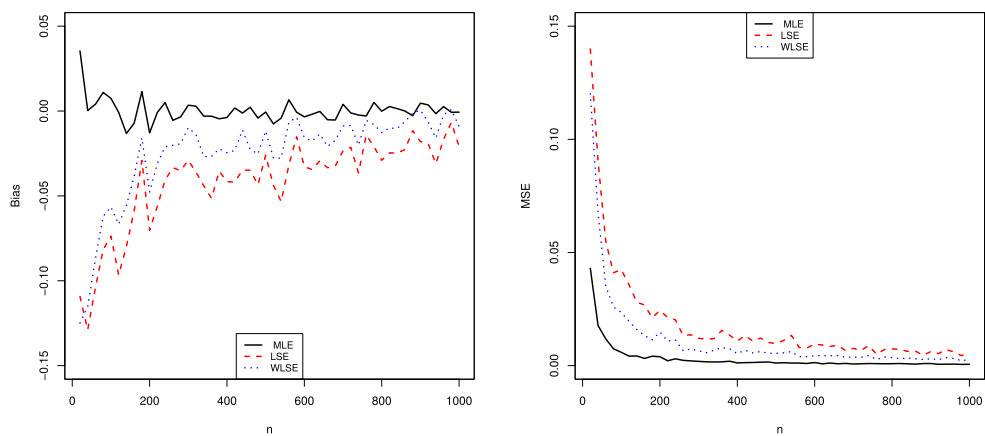


Figure 8. The simulation results of the σ parameters.

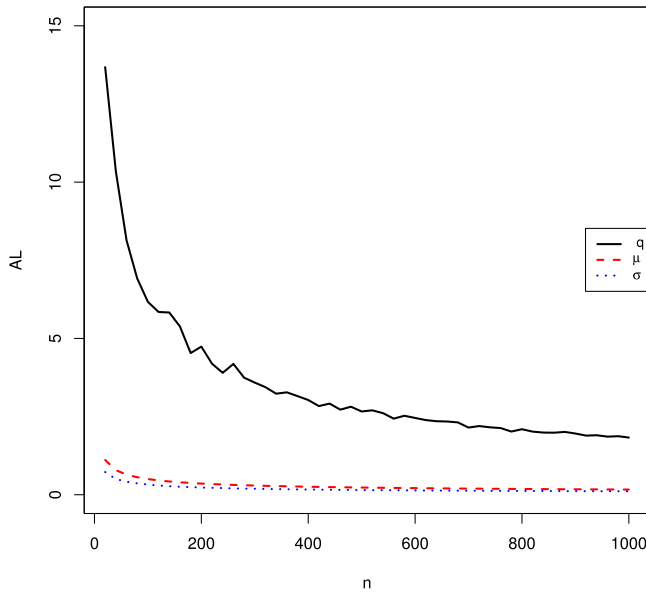


Figure 9. The estimated ALs for the selected parameter vector.

$(s_{\hat{q}_i}, s_{\hat{\mu}_i}, s_{\hat{\sigma}_i})$ for $i = 1, \dots, N$, are evaluated by inverting the observed information matrix. The estimated ALs are given by

$$AL_{\epsilon}(n) = \frac{3.919928}{N} \sum_{i=1}^N s_{\hat{\epsilon}_i}.$$

Figure 9 displays the simulation results for the above measure. As seen from Figure 9, as expected, when the sample size increases the AL decreases for each parameter. The simulation results verify the consistency property of MLEs.

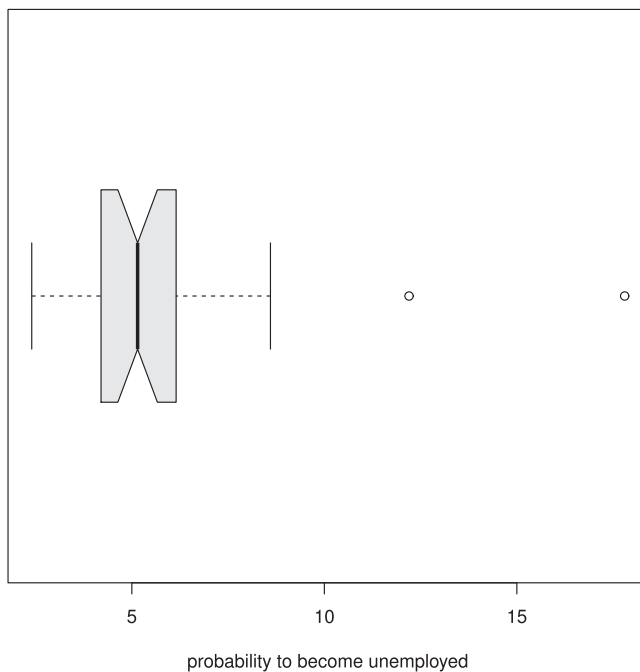
5. Data analysis

The Better Life Index (BLI) data set, measured in the year 2015, is used to demonstrate the usefulness of LSI distribution model. The data set can be found in <https://stats.oecd.org/index.aspx?DataSetCode=BLI2015>. The BLI data set consists of 11 indicator and 24 variables and it is used to classify the OECD (Organisation for Economic Co-operation and Development) countries as well as Brazil and Russia. Here, we use an indicator that is entitled Job security as data set. This indicator presents the probability to become unemployed and it is calculated as the number of people who were unemployed in 2013 but were employed in 2012 over the total number of employed in 2012.

We give the summary statistics of the data set in Table 2. The data is right skewed and has the large kurtosis. Figure 10 presents the box plot of the data set. This figure shows that since the observations, which are 0.122 and 0.178, may be outlier observations for the data set, it can be needed the heavy-tailed (thicker tail) distribution to model this data set. Therefore, we will analyze this data set without outlier observations before we re-analyze this data set with outlier observations to see the necessity of the thick-tailed distribution.

Table 2. Some summary statistics of the data set.

Minimum	Mean	Median	Maximum	Variance	Skewness	Kurtosis	<i>n</i>
0.0240	0.0567	0.0515	0.1780	0.0007	2.7117	12.0173	36

**Figure 10.** The box plot of the data set.

Under MLE method, we now fit the *LSI* distribution to this data set and compare it with some distributions, which are defined on $(0,1)$ interval. These competitor distributions are (with their pdfs for $0 < x < 1$):

- Beta distribution:

$$f_{\text{Beta}}(x, \mu, \sigma) = \frac{1}{B(\mu, \sigma)} x^{\mu-1} (1-x)^{\sigma-1}, \quad \mu, \sigma > 0,$$

where $B(\mu, \sigma)$ is the beta function.

- Kw distribution:

$$f_{\text{Kw}}(x, \mu, \sigma) = \mu \sigma x^{\mu-1} (1-x^{\mu})^{\sigma-1}, \quad \mu, \sigma > 0.$$

- ETL distribution:

$$f_{\text{ETL}}(x, \mu, \sigma) = 2\mu\sigma (1-x) [x(2-x)]^{\mu-1} [1-x^{\mu}(2-x)^{\mu}]^{\sigma-1}, \quad \mu, \sigma > 0.$$

- McA distribution:

$$f_{McA}(x, q, \mu, \sigma) = \frac{q}{\pi B(\mu, \sigma) \sqrt{x - x^2}} \left(\frac{2}{\pi} \arcsin \sqrt{x} \right)^{q\mu-1} \times \left(1 - \left(\frac{2}{\pi} \arcsin \sqrt{x} \right)^q \right)^{\sigma-1}, \quad q, \mu, \sigma > 0.$$

- UIG distribution:

$$f_{UIG}(x, \mu, \sigma) = \sqrt{\frac{\mu}{2\pi}} \frac{1}{x(-\log x)^{3/2}} \exp \left[\frac{\mu}{2\sigma^2 \log x} (\log x + \sigma)^2 \right], \quad \mu, \sigma > 0.$$

- Johnson S_B and logit normal distributions.

To determine the best model, we also compute the estimated log-likelihood values $\hat{\ell}$, Akaike Information Criteria (AIC), Bayesian information criterion (BIC), Kolmogorov–Smirnov (KS), Cramer–von-Mises, (W^*) and Anderson–Darling (A^*) goodness-of-fit statistics for all distribution models. In general, it can be chosen as the best model the one which has the smaller the values of the AIC, BIC, KS, W^* and A^* statistics and the larger the values of $\hat{\ell}$ and p -value of the goodness-of-statistics. All computations are performed by the **maxLik** and **gofest** routines in the R program. The details are given below.

Tables 3 and Figure 11 respectively show analyzing results and estimated plots based on above distribution models for the data set without the outlier observations. We can see from the results that the McA distribution may be preferred as the best model in terms of all comparing criteria. The fits of the LSI , Johnson S_B and logit normal distributions are same.

To see the applicability and efficiency of the LSI distribution model for the outlier situation, we re-analyze the complete data set for all distributions. We give the estimates and

Table 3. MLEs, standard erros of the estimates (in parentheses), $\hat{\ell}$ and goodness-of-fits statistics for the data set with without outlier observations (p -value is given in [·]).

Model	\hat{q}	$\hat{\mu}$	$\hat{\sigma}$	$\hat{\ell}$	AIC	BIC	A^*	W^*	KS
LSI	152.1421 (8.3888)	−2.9729 (0.0487)	0.2822 (0.0342)	99.1352	−192.2705	−187.6914	0.4304	0.0655	0.0987 [0.8949]
Beta		13.8730 (0.5031)	261.9929 (5.1197)	99.7729	−195.5457	−192.4930	0.2907	0.0422	0.0831 [0.9731]
Kw		4.3364 (0.7443)	285634.1 8.4216	99.8919	−195.7838	−192.7311	0.1823	0.0235	0.0865 [0.9612]
ETL		4.4632 (0.0748)	21252.81 (8.3886)	99.9258	−195.8516	−192.7989	0.1785	0.0229	0.0855 [0.9650]
McA	2.3010 (0.9887)	59762.03 (2.2106)	5.3175 (0.2670)	100.1631	−194.3262	−189.7471	0.1719	0.0217	0.0729 [0.9936]
Johnson S_B		10.4638 (1.2755)	3.5196 (0.4252)	99.1352	−194.2705	−191.2178	0.4307	0.0656	0.0988 [0.8944]
LN		−2.9729 (0.0487)	0.2841 (0.0344)	99.1352	−194.2705	−191.2178	0.4308	0.0657	0.0988 [0.8941]
UIG		0.6606 (0.1602)	0.0503 (0.0024)	98.9600	−193.9200	−190.8673	0.4820	0.0762	0.1044 [0.8523]

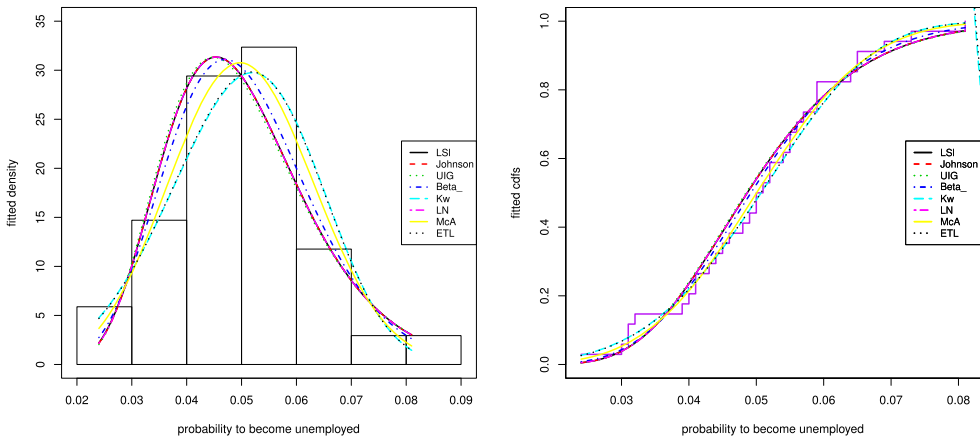


Figure 11. Estimated densities and cdfs for the data set without the outlier points.

Table 4. MLEs, standard errors of the estimates (in parentheses), $\hat{\ell}$ and goodness-of-fits statistics for the data set with outlier observations (p -value is given in [·]).

Model	\hat{q}	$\hat{\mu}$	$\hat{\sigma}$	$\hat{\ell}$	AIC	BIC	A^*	W^*	KS
LSI	2.2687 (1.0992)	-2.9254 (0.0556)	0.2117 (0.0613)	91.8883	-177.7767	-173.0262	0.1870	0.0225	0.0586 [0.9997]
Beta		5.8569 (0.5166)	97.1458 (6.2564)	86.9760	-169.9519	-166.7848	1.1152	0.1768	0.1636 [0.2903]
Kw		2.1577 (0.0648)	373.3878 8.4525	82.0487	-160.0975	-156.9305	2.2041	0.3651	0.1916 [0.1422]
ETL		2.2617 (0.0883)	111.6719 (6.2017)	82.4756	-160.9512	-157.7841	2.1134	0.3493	0.1898 [0.1495]
McA	203.8573 (0.4951)	90.9007 (0.7086)	0.1927 (0.0038)	88.5837	-171.1707	-166.4202	0.8289	0.1266	0.1443 [0.4412]
Johnson S_B		7.1149 (0.8440)	2.4608 (0.2864)	89.6573	-175.3146	-172.1476	0.6666	0.1008	0.1322 [0.5554]
logit normal		-2.8912 (0.0677)	0.4064 (0.0479)	89.6573	-175.3146	-172.1476	0.6668	0.1008	0.1321 [0.5553]
UIG		0.3608 (0.0850)	0.0567 (0.0037)	89.8754	-175.7507	-172.5837	0.6775	0.1064	0.1346 [0.5317]

the values of goodness-of-fits statistics in Table 4. When we see this Table, the *LSI* distribution model can be chosen as the best model since it has the smallest values of the AIC, BIC, KS, W^* and A^* statistics and the largest values of $\hat{\ell}$. Further, it has the largest p -value of the KS statistics among all models. While the data set does not contain outlier observations, we have chosen the McA model as the best model. Whereas we chose the *LSI* model as the best model, when the data set contains the outlier observations. Further, we sketch all fitted densities, cdfs, tail plot and PP plot of fitted *LSI* distribution for the complete data set in Figure 11. We observe that the *LSI* fit is better than the other fits. Furthermore proposed model has ensured to successfully captured the large kurtosis, skewness and heavy tailedness properties for the complete data set which has outliers. Consequently, the *LSI* distribution provides a better fit than above all distributions when the data set has outliers.

The value of the LR statistic for testing the hypothesis $H_0 : LN$ against $H_1 : LSI$ is $w = 4.4621$ with 0.0347 p -value ($p < 0.05$). Hence, one can reject the null hypotheses in

favor of the *LSI* distribution at %5 significance level. Thus, the *LSI* distribution provides a better representation of the complete data set than logit normal distribution since the additional parameter of the *LSI* distribution is essential.

6. Future work and conclusion

Future research would be a flexible and heavy-tailed distribution family based on the *LSI* distribution. A method of generating families of distributions is to combine with $F(H)$ structure which have the cdf as the value of the cdf of the distribution F whose range is the $(0,1)$ interval H . With this idea, the cdf and pdf of the new flexible family can be defined by

$$F(x, q, \mu, \sigma, \xi) = q \int_0^1 t^{q-1} \Phi \left(t \left[\frac{\log \left(\frac{G(x; \xi)}{1 - G(x; \xi)} \right) - \mu}{\sigma} \right] \right) dt$$

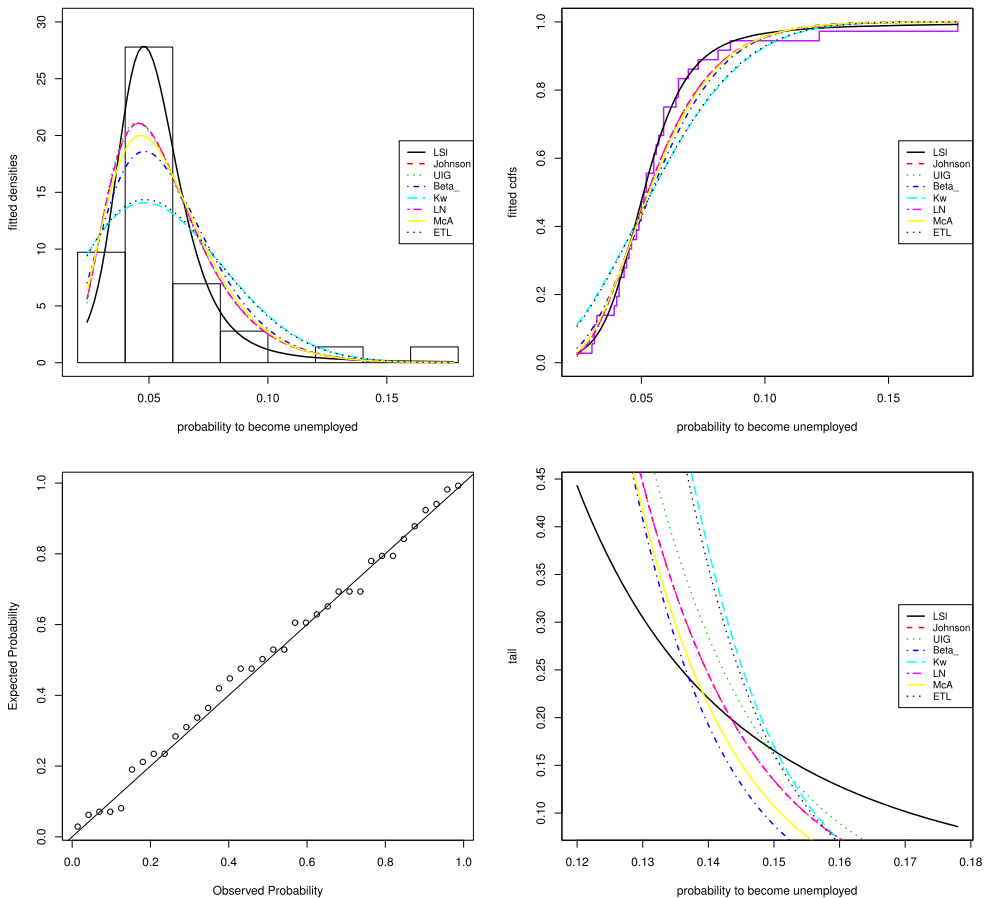


Figure 12. Estimated densities, cdfs, distribution tails and PP plot of the *LSI* distribution for the data set with the outlier points.

and

$$f(x, q, \mu, \sigma, \xi) = \frac{q g(x; \xi)}{G(x; \xi) [1 - G(x; \xi)] \sigma} \int_0^1 t^q \phi \left(t \left[\frac{\log \left(\frac{G(x; \xi)}{1 - G(x; \xi)} \right) - \mu}{\sigma} \right] \right) dt,$$

respectively, where $x \in \mathbb{R}$, $q, \sigma > 0$ and $-\infty < \mu < \infty$ are additional parameters, $G(x; \xi)$ is the any baseline cdf with parameter vector ξ and $g(x; \xi)$ is the corresponding pdf of $G(x; \xi)$.

This distribution family will generate strong competitors of the heavy-tailed distributions.

For the conclusions, we introduced and studied a new heavy-tailed bounded distribution, defined on $(0,1)$ interval, using slash distribution and logit function structures. We investigated the general structural properties of the new distribution. The model parameters were estimated by maximum likelihood, least-square, and weighted least-square methods. A simulation study was performed to illustrate the performances of estimators. Its usefulness on data modeling was shown via an application to the real data set. In summary, the proposed model can be an alternative to the classical bounded distributions available in the statistical literature to model rates and proportions.

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References

- [1] E. Altun, *The log-weighted exponential regression model: alternative to the beta regression model*, Commun. Stat.-Theory Methods (2019). doi:10.1080/03610926.2019.1664586.
- [2] E. Altun and G.G. Hamedani, *The log-xgamma distribution with inference and application*, J. de la Société Française de Stat. 159 (2018), pp. 40–55.
- [3] B.C. Arnold and R.A. Groeneveld, *Some properties of the arcsine distribution*, J. Am. Stat. Assoc. 75 (1980), pp. 173–175.
- [4] O. Arslan, *An alternative multivariate skew-slash distribution*, Stat. Probab. Lett. 78 (2008), pp. 2756–2761.
- [5] O. Arslan and A.I. Genç, *A generalization of the multivariate slash distribution*, J. Stat. Plan. Inference. 139 (2009), pp. 1164–1170.
- [6] M. Bebbington, C.D. Lai, D.N.P. Murthy, and R. Zitikis, *Modelling N-and W-shaped hazard rate functions without mixing distributions*, Proc. Inst. Mech. Eng. O: J. Risk Reliab. 223 (2009), pp. 59–69.
- [7] R.P. Brent, *Some efficient algorithms for solving systems of nonlinear equations*, SIAM. J. Numer. Anal. 10 (1973), pp. 327–344.
- [8] Y.M. Bulut, J. Ahad, F.Z. Dogru, and O. Arslan, *The generalized half-t distribution*, Stat. Interface 10 (2017), pp. 727–734.

- [9] Y.M. Bulut and O. Arslan, *Matrix variate slash distribution*, J. Multivar. Anal. 137 (2015), pp. 173–178.
- [10] P.C. Consul and G.C. Jain, *On the log-gamma distribution and its properties*, Stat. Papers 12 (1971), pp. 100–106.
- [11] G.M. Cordeiro and A.J. Lemonte, *The McDonald arcsine distribution: A new model to proportional data*, Statistics 48 (2014), pp. 182–199.
- [12] S. Dey, S.D. Kumar, P.L. Ramos, and F. Louzada, *Exponentiated Chen distribution: Properties and estimation*, Commun. Stat.-Simul. Comput. 46 (2017), pp. 8118–8139.
- [13] P. Frederic and F. Lad, *Two moments of the logit normal distribution*, Commun. Stat.-Simul. Comput. 37 (2008), pp. 1263–1269.
- [14] A.A. Genç, M.Ç Korkmaz, and C. Kus, *The beta Moyal-slash distribution*, J. Selcuk Univ. Natl. Appl. Sci. 3 (2014), pp. 88–104.
- [15] A.I. Genç, *A generalization of the univariate slash by a scale-mixed exponential power distribution*, Commun. Stat.-Simul. Comput. 36 (2007), pp. 937–947.
- [16] A.I. Genç, *A skew extension of the slash distribution via beta-normal distribution*, Stat. Papers 54 (2013), pp. 427–442.
- [17] M.E. Ghitany, J. Mazucheli, A.F.B. Menezes, and F. Alqallaf, *The unit-inverse Gaussian distribution: A new alternative to two-parameter distributions on the unit interval*, Commun. Stat.-Theory Methods 48 (2019), pp. 3423–3438.
- [18] Y.M. Gómez, H. Bolfarine, and H.W. Gómez, *Gumbel distribution with heavy tails and applications to environmental data*, Math. Comput. Simul. 157 (2019), pp. 115–129.
- [19] E. Gómez-Déniz, A.S. Miguel, and E. Calderín-Ojeda, *The log-Lindley distribution as an alternative to the beta regression model with applications in insurance*, Insur. Math. Econ. 54 (2014), pp. 49–57.
- [20] E. Gómez-Déniz and J.M. Sarabia, *A family of generalised beta distributions: Properties and applications*, Ann. Data Sci. 5 (2018), pp. 401–420.
- [21] Y.A. Iriarte, H.W. Gómez, H. Varela, and H. Bolfarine, *Slashed rayleigh distribution*, Rev. Colomb. de Estadística 38 (2015), pp. 31–44.
- [22] M. Jamshidian, *A note on parameter and standard error estimation in adaptive robust regression*, J. Stat. Comput. Simul. 71 (2001), pp. 11–27.
- [23] N.L. Johnson, *Systems of frequency curves generated by methods of translation*, Biometrika 36 (1949), pp. 149–176.
- [24] N.L. Johnson, *Systems of frequency curves derived from the first law of Laplace*, Trabajos de Estadística y de Investigación Operativa 5 (1955), pp. 283–291.
- [25] D.N. Kashid and S.R. Kulkarni, *Subset selection in multiple linear regression with heavy tailed error distribution*, J. Stat. Comput. Simul. 73 (2003), pp. 791–805.
- [26] M.Ç. Korkmaz, *A generalized skew slash distribution via gamma-normal distribution*, Commun. Stat.-Simul. Comput. 46 (2017), pp. 1647–1660.
- [27] M.Ç. Korkmaz and A.I. Genç, *A new generalized two-sided class of distributions with an emphasis on two-sided generalized normal distribution*, Commun. Stat.-Simul. Comput. 46 (2017), pp. 1441–1460.
- [28] P. Kumaraswamy, *A generalized probability density function for double-bounded random processes*, J. Hydrol. (Amst) 46 (1980), pp. 79–88.
- [29] F. Louzada, P.L. Ramos, and G.S.C. Perdoná, *Different estimation procedures for the parameters of the extended exponential geometric distribution for medical data*, Comput. Math. Methods. Med. 2016 (2016), pp. 1–12.
- [30] J. Mazucheli, A.F.B. Menezes, and S. Chakraborty, *On the one parameter unit-Lindley distribution and its associated regression model for proportion data*, J. Appl. Stat. 46 (2019), pp. 700–714.
- [31] J. Mazucheli, A.F.B. Menezes, and S. Dey, *The unit-Birnbaum-Saunders distribution with applications*, Chil. J. Stat. 9 (2018), pp. 47–57.
- [32] R. Mead, *A generalised logit-normal distribution*, Biometrics 21 (1965), pp. 721–732.
- [33] J.B. McDonald, *Some generalized functions for the size distribution of income*, Econometrica 52 (1984), pp. 647–663.

- [34] S. Morgenthaler, *Robust confidence intervals for a location parameter: The configural approach*, J. Am. Stat. Assoc. 81 (1986), pp. 518–525.
- [35] N.M. Olmos, H. Varela, H.W. Gómez, and H. Bolfarine, *An extension of the half-normal distribution*, Stat. Papers 53 (2012), pp. 875–886.
- [36] P. Pinson, *Very short term probabilistic forecasting of wind power with generalized logit-normal distributions*, J. R. Stat. Soc.: Ser. C (Appl. Stat.) 61 (2012), pp. 555–576.
- [37] A. Pourdarvish, S.M.T.K. Mirmostafae, and K. Naderi, *The exponentiated Topp-Leone distribution: Properties and application*, J. Appl. Environ. Biolog. Sci. 5 (2015), pp. 251–256.
- [38] J. Reyes, H.W. Gómez, and H. Bolfarine, *Modified slash distribution*, Stat. 47 (2013), pp. 929–941.
- [39] J. Reyes, Y.A. Iriarte, P. Jodrá, and H.W. Gómez, *The slash Lindley-Weibull distribution*, Methodol. Comput. Appl. Probab. 21 (2019), pp. 235–251.
- [40] G.C. Rodrigues, F. Louzada, and P.L. Ramos, *Poisson-exponential distribution: Different methods of estimation*, J. Appl. Stat. 45 (2018), pp. 128–144.
- [41] W.H. Rogers and J.W. Tukey, *Understanding some long-tailed symmetrical distributions*, Stat. Neerl. 26 (1972), pp. 211–226.
- [42] P.R. Tadikamalla and N.L. Johnson, *Systems of frequency curves generated by transformations of logistic variables*, Biometrika 69 (1982), pp. 461–465.
- [43] C.W. Topp and F.C. Leone, *A family of J-shaped frequency functions*, J. Am. Stat. Assoc. 50 (1955), pp. 209–219.
- [44] R.J. Van Dorp and S. Kotz, *The standard two-sided power distribution and its properties: With applications in financial engineering*, Am. Stat. 56 (2002), pp. 90–99.
- [45] J. Wang and M.G. Genton, *The multivariate skew-slash distribution*, J. Stat. Plan. Inference. 136 (2006), pp. 209–220.

Appendices

In here, the components of the score vectors, which belong to three estimation methods in Section 3, are given.

Appendix 1. The components of score vector for MLE method

The \hat{q}_{MLE} , $\hat{\mu}_{MLE}$ and $\hat{\sigma}_{MLE}$ can be obtained as the solution of the following system of equations:

$$U_q = \frac{n}{q} + \sum_{i=1}^n \frac{\int_0^1 t^q \phi(t u_i) \log t \, dt}{\int_0^1 t^q \phi(t u_i) \, dt} = 0,$$

$$U_\mu = \sum_{i=1}^n \frac{\int_0^1 t^{q+2} (u_i/\sigma) \phi(t u_i) \, dt}{\int_0^1 t^q \phi(t u_i) \, dt} = 0$$

and

$$U_\sigma = -\frac{n}{\sigma} + \sum_{i=1}^n \frac{\int_0^1 t^{q+2} (u_i^2/\sigma) \phi(t u_i) \, dt}{\int_0^1 t^q \phi(t u_i) \, dt} = 0.$$

Appendix 2. The components of score vector for LSE method

The \hat{q}_{LSE} , $\hat{\mu}_{LSE}$ and $\hat{\sigma}_{LSE}$ can be obtained as the solution of the following system of equations:

$$\frac{\partial QLSE(\Xi)}{\partial q} = \sum_{i=1}^n F'_q(x_{(i)}, q, \mu, \sigma) \left(F(x_{(i)}, q, \mu, \sigma) - \frac{i}{n+1} \right) = 0,$$

$$\frac{\partial QLSE(\Xi)}{\partial \mu} = \sum_{i=1}^n F'_\mu(x_{(i)}, q, \mu, \sigma) \left(F(x_{(i)}, q, \mu, \sigma) - \frac{i}{n+1} \right) = 0$$

and

$$\frac{\partial QLSE(\Xi)}{\partial \sigma} = \sum_{i=1}^n F'_\sigma(x_{(i)}, q, \mu, \sigma) \left(F(x_{(i)}, q, \mu, \sigma) - \frac{i}{n+1} \right) = 0,$$

where

$$F'_q(x, q, \mu, \sigma) = q^{-1} F(x, q, \mu, \sigma) + q \int_0^1 t^{q-1} \Phi \left(\frac{t \left(\log \left(\frac{x}{1-x} \right) - \mu \right)}{\sigma} \right) \log t \, dt,$$

$$F'_\mu(x, q, \mu, \sigma) = x(x-1) f(x, q, \mu, \sigma)$$

and

$$F'_\sigma(x, q, \mu, \sigma) = \sigma^{-1} x(x-1) \left(\log \left(\frac{x}{1-x} \right) - \mu \right) f(x, q, \mu, \sigma).$$

Appendix 3. The components of score vector for WLSE method

The \hat{q}_{WLSE} , $\hat{\mu}_{WLSE}$ and $\hat{\sigma}_{WLSE}$ can be obtained as the solution of the following system of equations:

$$\frac{\partial QWLSE(\Xi)}{\partial q} = (n+2)(n+1)^2 \sum_{i=1}^n \left(F(x_{(i)}, q, \mu, \sigma) - \frac{i}{n+1} \right) \frac{F'_q(x_{(i)}, q, \mu, \sigma)}{i(n-i+1)} = 0,$$

$$\frac{\partial QWLSE(\Xi)}{\partial \mu} = (n+2)(n+1)^2 \sum_{i=1}^n \left(F(x_{(i)}, q, \mu, \sigma) - \frac{i}{n+1} \right) \frac{F'_\mu(x_{(i)}, q, \mu, \sigma)}{i(n-i+1)} = 0$$

and

$$\frac{\partial QWLSE(\Xi)}{\partial \sigma} = (n+2)(n+1)^2 \sum_{i=1}^n \left(F(x_{(i)}, q, \mu, \sigma) - \frac{i}{n+1} \right) \frac{F'_\sigma(x_{(i)}, q, \mu, \sigma)}{i(n-i+1)} = 0,$$

where $F'_q(x_{(i)}, q, \mu, \sigma)$, $F'_\mu(x_{(i)}, q, \mu, \sigma)$ and $F'_\sigma(x_{(i)}, q, \mu, \sigma)$ are defined above.

As it can be seen, since all the above estimating equations contain non-linear functions, it is not possible to obtain explicit forms of the MLEs, LSEs, and WLSEs directly. Therefore, they have to be solved by using numerical methods such as the Newton–Raphson and quasi-Newton algorithms (see [12,29,40]).